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MAXIMAL SEPARABLE SUBFIELDS OF BOUNDED CODEGREE

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ABSTRACT. Let L be a function field in $n > 0$ variables over a field K of characteristic $p \neq 0$. An intermediate field S is maximal separable if S is separable over K and every subfield of L which properly contains S is inseparable over K. This paper examines when $\{[L:S] \mid S \text{ is maximal separable}\}$ is bounded. The main result states that this set is bounded if and only if there is an integer c such that any intermediate field L_1 over which L is purely inseparable and $[L:L_1] > p^c$ must be separable over K. Examples are also given where the above bound is p^{n+1} for any $n \geqslant 1$.

Let L be a function field in n ($n > 0$) variables over a field K of characteristic $p \neq 0$. An intermediate field S is maximal separable if S is separable over K and every subfield of L which properly contains S is inseparable over K. It is clear that L is purely inseparable and finite dimensional over any maximal separable S. This paper is concerned with $\{[L:S] \mid S \text{ is maximal separable}\}$. Such an S is distinguished if $L \subseteq K^{p^{-\infty}}(S)$, that is, L is contained in a field obtained from S by adjoining only roots of elements of K. Every L/K has distinguished subfields and moreover, S' is distinguished if and only if $[L:S'] = \min\{[L:S] \mid S \text{ is maximal separable}\}$ **[8]**. If this minimum is p^r, then r is called the order of inseparability of L/K, denoted $\operatorname{inor}(L/K)$. **[2]** examined the question of when every maximal separable subfield of L/K is distinguished, i.e., $\{[L:S] \mid S \text{ is maximal separable}\} = \{p^r\}$. Recently Heerema, **[7]**, examined the question of when $\{[L:S] \mid S \text{ is maximal separable}\}$ is bounded. He showed, for the case where L is of transcendence degree 1 over K, that this set is bounded if and only if the algebraic closure of K in L is separable over K. This paper continues the investigations begun in **[7]**.

If $\{[L:S] \mid S \text{ is maximal separable}\}$ is bounded, then any intermediate field L_1, over which L is not algebraic, must be separable over K (Corollary 6). In some special cases, the converse of this result is also true, and we conjecture it is true in general. The main result, Theorem 10, gives a characterization of when $\{[L:S] \mid S \text{ is maximal separable}\}$ is bounded. We also give examples of extensions where p^{n+1} is the bound for $\{[L:S] \mid S \text{ is maximal separable}\}$, $n \geqslant 1$.

We will need the following notions. $\operatorname{Insep}(L/K) = \log_p[L:K(L^p)]$—the transcendence degree of L/K. Since $\operatorname{insep}(L/K) = 0$ if and only if L is separable over K, $\operatorname{insep}(L/K)$ is a measure of the inseparability of L/K. $\operatorname{Inex}(L/K) = \min\{r \mid K(L^{p^r}) \text{ is separable over } K\}$. If L_1 is an intermediate field of L/K, then $\operatorname{inor}(L/K) \geqslant \operatorname{inor}(L_1/K)$, and we have equality if and only if L^{p^n} and $K(L_1^{p^n})$ are linearly disjoint over $L_1^{p^n}$ for all n **[1**, p. 656]. If $\operatorname{inor}(L/K) = \operatorname{inor}(L_1/K)$, then L_1

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is called a form of L/K. The fields $K(L^{(n)}) = \{x \in L \mid x^{p^t} \in K(L^{p^{t+n}})$ for some $t \geq 0\}$ were first introduced in [6]. For $n \geq \text{inex}(L/K)$, $K(L^{(n)})$ has $K(L^{p^n})$ as a maximal separable subfield.

PROPOSITION 1. *Let $\{L_n \mid 1 \leq n < \infty\}$ be a descending chain of intermediate fields of L/K. Then $\bigcap L_n$ is separable over K if and only if there exits $n_0 < \infty$ such that L_{n_0} is separable over K.*

PROOF. Inor$(L_1/K) \geq$ inor$(L_2/K) \geq \cdots$ by [1, Theorem 1.2, p. 656]. Since this is a nonincreasing sequence of nonnegative integers, it must eventually become constant. Let n_0 be such that inor$(L_{n_0}/K) =$ inor$(L_{n_0+1}/K) = \cdots$. Then L_{n_0+j}/K is a form of L_{n_0}/K for all $j \geq 0$. Hence $\bigcap L_n/K$ is a form of L_{n_0}/K by the proof of [1, Theorem 1.4, p. 657]. Thus $\bigcap L_n$ is separable over K if and only if L_{n_0} is separable over K.

COROLLARY 2. *L/K has an infinite descending proper chain of inseparable intermediate fields if and only if there is an intermediate field L_1 which is inseparable over K and over which L is not algebraic.*

PROOF. This follows from Proposition 1 and the fact that a finitely generated field extension with an infinite proper chain of intermediate fields cannot be an algebraic field extension.

Let $\bar{K}$ denote the algebraic closure of K in L. The following result is [6, Corollary 6, p. 289], however Proposition 1 gives a simple proof.

COROLLARY 3. *$\bar{K}/K$ is separable if and only if $K(L^{(n)}) = K(L^{p^n})$ for some n.*

PROOF. $K(L^{(n)}) \supseteq K(L^{p^n})$ and has $K(L^{p^n})$ as a maximal separable extension of K in $K(L^{(n)})$. Thus $K(L^{(n)}) = K(L^{p^n})$ if and only if $K(L^{(n)})$ is separable over K. Since $\bigcap K(L^{(n)}) = \bar{K}$ [6, Theorem 5, p. 289], the result follows from Proposition 1.

Recall that a separable extension of S of K is maximal separable extension of K in L if and only if L is purely inseparable over S and $L^p \cap S \subseteq K(S^p)$ [5, Lemma 1.2, p. 46]. In particular, if a relative p-basis for S over K remains p-independent in L, then clearly $L^p \cap S \subseteq K(S^p)$. If $L = L_1(x)$ where x is transcendental over L_1, then L is said to be ruled over L_1.

THEOREM 4. *If L is ruled over an intermediate field L_1 and L_1 is inseparable over K, then L has maximal separable subfields of arbitrarily large codegree.*

PROOF. Let $L = L_1(t)$ and let $\{z_1, \ldots, z_r, w_1, \ldots, w_s\}$ be a relative p-basis of L_1/K where $\{z_1, \ldots, z_r\}$ is a separating transcendence basis of a distinguished subfield D_1 of L_1/K Let $S = D_1(w_1 + t^{p^n})$. Since t is transcendental over L_1, $w_1 + t^{p^n}$ is transcendental over D_1 and hence S is separable over K. Since w_1 is purely inseparable over D_1, t, and hence L, is purely inseparable over S. Since $\{z_1, \ldots, z_r, w_1 + t^{p^n}\}$ is a relative p-basis of S over K which remains p-independent in L, S is a maximal separable extension of K in L by the comments preceeding Theorem 4. Finally, since $S(L_1) = L_1(t^{p^n})$, $p^n = [L : S(L_1)] \geq [L : S]$, and hence we can find maximal separable subfields of arbitrarily high codegree.

REFEREE'S LEMMA 5 [7, Footnote, p. 354]. *If L/L_1 is finite dimensional and L_1/K has maximal separable subfields of arbitrarily high codegree, then so does L.*

PROOF. Let S_1 be a maximal separable subfield of L_1 of high codegree. Then L_1/S_1 is purely inseparable and L_1 has at most $\log_p[L_1 : K(L_1^p)]$ generators over S_1. But $\log_p[L_1 : K(L_1^p)] \leq \log_p[L : K(L^p)] =$ some fixed constant [8, Lemma 1, p. 111]. There must be an element, say b, of large exponent, say n, over S_1. Let S be a maximal separable extension of K in L containing S_1. S exists by Zorn's Lemma. Then $S \cap L_1 = S_1$. Thus b is of exponent n over S, and hence $[L : S] \geq p^n$. Thus L has maximal separable subfields of large codegree.

COROLLARY 6. *If $\{[L : S] \mid S$ is a maximal separable extension of K in $L\}$ is bounded, then any intermediate field L_1 over which L is not algebraic must be separable over K.*

PROOF. Apply Theorem 4 and Lemma 5.

An intermediate field L_1 of L/K has the same inseparability over K as does L if and only if L^p and $K(L_1^p)$ are linearly disjoint over L_1^p [8, Lemma 1, p. 111]. The proof of [1, Theorem 1.4, p. 657] shows there is a unique minimal intermediate field L_I which has insep$(L_I/K) =$ insep(L/K). (L_I is merely the intersection of all subfields of L_1 with insep$(L_1/K) =$ insep(L/K).)

THEOREM 7. *Assume* insep$(L/K) = 1$. *The following are equivalent.*

(1) $\{[L : S] \mid S$ *is maximal separable extension of K in* $L\}$ *is bounded.*

(2) *Any intermediate field L_1 over which L is not algebraic must be separable over K.*

(3) *L is algebraic over L_I.*

PROOF. (1) implies (2) is Corollary 6. Assume (2). Since L_I is inseparable over K, L must be algebraic over L_I. Assume (3). Let S be maximal separable and let $b \in L \setminus S$ with $b^p \in S$. Then $S(b)$ is inseparable over K, and hence $S(b) \supseteq L_I$. Thus $[L : S] \leq p \cdot [L : L_I]$.

EXAMPLE 8. We give a family of extensions L_n/K where inor$(L_n/K) = 1$ and the bound of Theorem 7 is exactly p^{n+1}. Let $L_n = K(x, z, uz^{p^n} + xv + w), K = P(u^p, v^p, w^p)$ where P is a perfect field of characteristic $p \neq 0$ and $\{x, z, u, v, w\}$ is algebraically independent over P. L_n has a subfield $L_{n_1} = K(x, z^{p^n}, uz^{p^n} + xv + w)$ which is separable algebraic over its irreducible form [2, Example 11, p. 190] and [2, Corollary 7, p. 188], call it L_I. Since inor$(L_n/K) =$ inor$(L_{n_1}/K) = 1$, L_I is the irreducible form of L_n/K. Clearly $[L_n : L_{n_1}] = p^n$. Let S be a maximal separable extension of K in L_n, and let $b \in L_n \setminus S$ with $b^p \in S$. Then inor$(S(b)/K) = 1$, and hence $S(b)$ must contain L_I. But L_{n_1} is separable algebraic over L_I, and hence is contained in $S(b)$ since L_n is purely inseparable over $S(b)$. Thus $[L_n : S(b)] \leq [L_n : L_{n_1}] = p^n$. Thus $[L_n : S] \leq p^{n+1}$. But $K(x, uz^{p^n} + xv + w)$ is a maximal separable extension of K in L_n (see the comments preceding Theorem 4) which is of codegree p^{n+1}. Thus the bound of Theorem 7 is precisely p^{n+1}.

THEOREM 9. *Assume* tr.d.$(L/K) = 1$. *The following are equivalent.*

(1) $\overline{K}/K$ *is separable.*

(2) *There is an integer* c *such that any intermediate field* L_1 *over which* L *is purely inseparable and* $[L: L_1] > p^c$ *must be separable over* K.

(3) $\{[L: S] \mid S$ *is maximal separable*$\}$ *is bounded.*

PROOF. Assume (1). The proof is by induction on inor(L/K). The result is trivially true for inor$(L/K) = 0$. Assume the result for inor$(L/K) \leqslant n-1$ and let inor$(L/K) = n$. Let L be purely inseparable over L_1 and suppose L_1 is inseparable over K. We need to show $[L: L_1]$ must be bounded for all such L_1. If L_1 contains a relatively p-independent element x of L/K, then L_1 contains the separable algebraic closure of $K(x)$, denoted $(K(x))\hat{}$, in L, since L/L_1 is purely inseparable. By the comments preceeding Theorem 4, $(K(x))\hat{}$ is a maximal separable extension of K in L. By [**7**, Theorem 1, p. 353], the degrees of L over its maximal separable intermediate fields is bounded, and since $L_1 \supseteq (K(x))\hat{}$, the degree of L over L_1 is bounded. If L_1 does not contain a relatively p-independent element, then $L_1 \subseteq K(L^p)$. By [**1**, Lemma 1.1, p. 656], inor$(K(L^p)/K) <$ inor(L/K) when inor$(L/K) > 0$. Thus by induction, the degree of $K(L^p)$ over L_1 is bounded, and hence also the degree of L over L_1. Clearly (2) implies (1) since $[L: \overline{K}(L^{p^{c+1}})] > p^c$. [**7**, Theorem 1, p. 353] shows (1) is equivalent to (3).

THEOREM 10. $\{[L: S] \mid S$ *is a maximal separable extension of* K *in* $L\}$ *is bounded if and only if there is an integer* c *such that any intermediate field* L_1 *over which* L *is purely inseparable and* $[L: L_1] > p^c$ *must be separable over* K.

PROOF. If S is maximal separable and $b \in L \backslash S$ with $b^p \in S$, then $S(b)$ is inseparable over K. Thus, the existence of c guarantees $[L: S(b)] \leqslant p^c$ and hence $[L: S] \leqslant p^{c+1}$. Now assume there is a bound on the codegrees of maximal separable subfields. We prove there is a c by induction on the transcendence degree of L/K. The case of transcendence degree 1 is Theorem 9. We assume there is a sequence $\{L_n\}$ of subfields of increasing codegree which are inseparable over K, with L/L_n purely inseparable, and get a contradiction.

Let x be a relatively p-independent element of L/K. Since the codegrees of maximal separable subfields is bounded, $\overline{K}$ is separable over K [**7**, Corollary 2, p. 354]. Thus x is transcendental over K. Since x is also relatively p-independent in L/K, any maximal separable extension of $K(x)$ in L is also a maximal separable extension of K in L. Thus there is a bound on the codegrees of maximal separable extensions of $K(x)$ in L, and hence by induction, there is a c for $L/K(x)$. Since x is transcendental over K, each $L_n(x)$ is inseparable over $K(x)$. Thus the set of codegrees of the $L_n(x)$ is bounded.

Let $[L: L_n] = p^{d_n}$ where d_n is an increasing sequence. Let $[L: L_n(x)] \leqslant p^{c_1}$ where c_1 is a constant. Consider the finite sequence $a_1, a_2, \ldots, a_{d_n}$ defined by $p^{a_r} = [L_n^{p^{-r}} \cap L: L_n^{p^{-r-1}} \cap L]$. Note that $[L: L_n] = p^{a_1 + a_2 + \cdots + a_{d_n}} = p^{d_n}$. Since $[L: L_n(x)] \leqslant p^{c_1}$ while $[L: L_n] = p^{d_n}$, x is of exponent at least $d_n - c_1$ over L_n. Thus, for $i = 1, \ldots, d_n - c_1$, $a_i \geqslant 1$, and at most the last c_1 of the a_i's are 0. Since $a_1 + \cdots + a_{d_n} = d_n$, at most c_1 of the a_i's can exceed 1. So, we have a finite sequence of increasing length

(d_n) with at most a fixed number of elements $(2c_1)$ different from 1. Thus we can find strings of consecutive 1's of increasing length, say w_n, which begin at least as sequence element $a_{d_n - w_n}$ for the sequence associated to L_n. Thus for $s = \text{inor}(L/K) + 1$, we can find, for large n, fields $L'_n \supseteq L_n$ such that $L'^{p^{-s}}_n \cap L$ is simple over L'_n. Rename this sequence as $\{L_n\}$.

We now want to see that $L_n^{p^{-s}} \cap L$ has the same order of inseparability over K as $L_n^{p^{-s+1}} \cap L$ has over K, that is $L_n^{p^{-s+1}} \cap L/K$ is a form of $L_n^{p^{-s}} \cap L/K$. We can write $L_n^{p^{-s}} \cap L = L_n(\theta)$ and $L_n^{p^{-s+1}} \cap L = L_n(\theta^p)$. Now, the increase in the order of inseparability of $L_n(\theta)$ depends upon $\min\{\max\{r \mid \theta^{p^s} \in K(L_n^{(r)})\}, s\}$ [**4**, Theorem 2, p. 374]. But this minimum must be $\max\{r \mid \theta^{p^s} \in K(L_n^{(r)})\} < s$ since $s > \text{inor}(L/K) \geqslant \text{inor}(L_n(\theta)/K)$. Since the increase in the order of inseparability of $L_n(\theta^p) = \min\{\max\{r \mid \theta^{p^s} \in K(L_n^{(r)})\}, s-1\}$, the increases will be the same, i.e., $L_n^{p^{-s+1}} \cap L/K$ is a form of $L_n^{p^{-s}} \cap L/K$.

By [**3**, Theorem 3.3], $L_n^{p^{-s+1}} \cap L/K$ has a distinguished subfield D_n not contained in any of $L_n^{p^{-s}} \cap L$. We claim D_n is a maximal separable subfield of L/K. Clearly D_n/K is separable and L/D_n is purely inseparable. Suppose $x^p \in D_n$, $x \notin D_n$. If $x \in L_n^{p^{-s+1}} \cap L$ then $x^p \in K(D_n^p)$ since D_n is maximal separable in $L_n^{p^{-s+1}} \cap L$. If $x \notin L_n^{p^{-s+1}} \cap L$, then x must be in $L_n^{p^{-s}} \cap L$ since $L_n^{p^{-s}} \cap L$ is simple over L_n by construction. If x^p were not in $K(D_n^p)$, then by a degree argument $D_n(x)$ would be distinguished for $L_n^{p^{-s}} \cap L/K$, a contradiction. Thus the sequence of $\{D_n\}$ is a set of maximal separable extensions of K of unbounded codegree, a contradiction. Thus there is a c as in the statement of the theorem.

It is clear that the existence of a c as in the previous theorem implies that any subfield L_1 over which L is not algebraic must be separable over K. The converse is true in the transcendence degree $(L/K) = 1$ case, Theorem 9, or the insep$(L/K) = 1$ case, Theorem 7. We conjecture that it is true in general.

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