MAXIMAL SEPARABLE SUBFIELDS OF BOUNDED CODEGREE

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Abstract. Let $L$ be a function field in $n > 0$ variables over a field $K$ of characteristic $p \neq 0$. An intermediate field $S$ is maximal separable if $S$ is separable over $K$ and every subfield of $L$ which properly contains $S$ is inseparable over $K$. This paper examines when $\{ [L : S] | S $ is maximal separable $\}$ is bounded. The main result states that this set is bounded if and only if there is an integer $c$ such that any intermediate field $L_1$ over which $L$ is purely inseparable and $[L : L_1] > p^c$ must be separable over $K$. Examples are also given where the above bound is $p^{n+1}$ for any $n > 1$.

Let $L$ be a function field in $n (n > 0)$ variables over a field $K$ of characteristic $p \neq 0$. An intermediate field $S$ is maximal separable if $S$ is separable over $K$ and every subfield of $L$ which properly contains $S$ is inseparable over $K$. It is clear that $L$ is purely inseparable and finite dimensional over any maximal separable $S$. This paper is concerned with $\{ [L : S] | S $ is maximal separable $\}$. Such an $S$ is distinguished if $L \subseteq K^{n+}(S)$, that is, $L$ is contained in a field obtained from $S$ by adjoining only roots of elements of $K$. Every $L/K$ has distinguished subfields and moreover, $S'$ is distinguished if and only if $[L : S'] = \min \{ [L : S] | S $ is maximal separable $\}$ [8]. If this minimum is $p^r$, then $r$ is called the order of inseparability of $L/K$, denoted $\text{inor}(L/K)$. [2] examined the question of when every maximal separable subfield of $L/K$ is distinguished, i.e., $\{ [L : S] | S $ is maximal separable $\} = \{ p^r \}$. Recently Heerema, [7], examined the question of when $\{ [L : S] | S $ is maximal separable $\}$ is bounded. He showed, for the case where $L$ is of transcendence degree 1 over $K$, that this set is bounded if and only if the algebraic closure of $K$ in $L$ is separable over $K$. This paper continues the investigations begun in [7].

If $\{ [L : S] | S $ is maximal separable $\}$ is bounded, then any intermediate field $L_1$, over which $L$ is not algebraic, must be separable over $K$ (Corollary 6). In some special cases, the converse of this result is also true, and we conjecture it is true in general. The main result, Theorem 10, gives a characterization of when $\{ [L : S] | S $ is maximal separable $\}$ is bounded. We also give examples of extensions where $p^{n+1}$ is the bound for $\{ [L : S] | S $ is maximal separable $\}$, $n \geq 1$.

We will need the following notions. $\text{Insep}(L/K) = \log_p [L : K(L^p)]$—the transcendence degree of $L/K$. Since $\text{insep}(L/K) = 0$ if and only if $L$ is separable over $K$, $\text{insep}(L/K)$ is a measure of the inseparability of $L/K$. $\text{Inex}(L/K) = \min \{ r | K(L^p) $ is separable over $K \}$. If $L_1$ is an intermediate field of $L/K$, then $\text{inor}(L/K) \geq \text{inor}(L_1/K)$, and we have equality if and only if $L^p$ and $K(L_1^p)$ are linearly disjoint over $L^{p^n}$ for all $n$. [1, p. 656]. If $\text{inor}(L/K) = \text{inor}(L_1/K)$, then $L_1$...
is called a form of $L/K$. The fields $K(L^{(n)}) = \{x \in L \mid x^{p^n} \in K(L^{p^n})\}$ for some $t \geq 0$ were first introduced in [6]. For $n \geq \text{inex}(L/K)$, $K(L^{(n)})$ has $K(L^{p^n})$ as a maximal separable subfield.

**Proposition 1.** Let $\{L_n \mid 1 \leq n < \infty\}$ be a descending chain of intermediate fields of $L/K$. Then $\bigcap L_n$ is separable over $K$ if and only if there exists $n_0 < \infty$ such that $L_{n_0}$ is separable over $K$.

**Proof.** Inor($L_1/K$) $\geq$ inor($L_2/K$) $\geq \cdots$ by [1, Theorem 1.2, p. 656]. Since this is a nonincreasing sequence of nonnegative integers, it must eventually become constant. Let $n_0$ be such that inor($L_{n_0}/K$) = inor($L_{n_0+1}/K$) = $\cdots$. Then $L_{n_0+j}/K$ is a form of $L_{n_0}/K$ for all $j \geq 0$. Hence $\bigcap L_n/K$ is a form of $L_{n_0}/K$ by the proof of [1, Theorem 1.4, p. 657]. Thus $\bigcap L_n$ is separable over $K$ if and only if $L_{n_0}$ is separable over $K$.

**Corollary 2.** $L/K$ has an infinite descending proper chain of inseparable intermediate fields if and only if there is an intermediate field $L_1$ which is inseparable over $K$ and over which $L$ is not algebraic.

**Proof.** This follows from Proposition 1 and the fact that a finitely generated field extension with an infinite proper chain of intermediate fields cannot be an algebraic field extension.

Let $\bar{K}$ denote the algebraic closure of $K$ in $L$. The following result is [6, Corollary 6, p. 289], however Proposition 1 gives a simple proof.

**Corollary 3.** $\bar{K}/K$ is separable if and only if $K(L^{(n)}) = K(L^{p^n})$ for some $n$.

**Proof.** $K(L^{(n)}) \supseteq K(L^{p^n})$ and has $K(L^{p^n})$ as a maximal separable extension of $K$ in $K(L^{(n)})$. Thus $K(L^{(n)}) = K(L^{p^n})$ if and only if $K(L^{(n)})$ is separable over $K$. Since $\bigcap K(L^{(n)}) = \bar{K}$ [6, Theorem 5, p. 289], the result follows from Proposition 1.

Recall that a separable extension of $S$ of $K$ is maximal separable extension of $K$ in $L$ if and only if $L$ is purely inseparable over $S$ and $L^p \cap S \subseteq K(S^p)$ [5, Lemma 1.2, p. 46]. In particular, if a relative $p$-basis for $S$ over $K$ remains $p$-independent in $L$, then clearly $L^p \cap S \subseteq K(S^p)$. If $L = L_1(x)$ where $x$ is transcendental over $L_1$, then $L$ is said to be ruled over $L_1$.

**Theorem 4.** If $L$ is ruled over an intermediate field $L_1$ and $L_1$ is inseparable over $K$, then $L$ has maximal separable subfields of arbitrarily large codegree.

**Proof.** Let $L = L_1(t)$ and let $\{z_1, \ldots, z_r, w_1, \ldots, w_s\}$ be a relative $p$-basis of $L_1/K$ where $\{z_1, \ldots, z_r\}$ is a separating transcendence basis of a distinguished subfield $D_1$ of $L_1/K$. Let $S = D_1(w_1 + t^p)$. Since $t$ is transcendental over $L_1$, $w_1 + t^p$ is transcendental over $D_1$ and hence $S$ is separable over $K$. Since $w_1$ is purely inseparable over $D_1$, $t$, and hence $L$, is purely inseparable over $S$. Since $\{z_1, \ldots, z_r, w_1 + t^p\}$ is a relative $p$-basis of $S$ over $K$ which remains $p$-independent in $L$, $S$ is a maximal separable extension of $K$ in $L$ by the comments preceding Theorem 4. Finally, since $S(L_1) = L_1(t^p)$, $p^n = \lbrack L : S(L_1) \rbrack \geq \lbrack L : S \rbrack$, and hence we can find maximal separable subfields of arbitrarily high codegree.
Referee’s Lemma 5 [7, Footnote, p. 354]. If $L/L_1$ is finite dimensional and $L_1/K$ has maximal separable subfields of arbitrarily high codegree, then so does $L$.

Proof. Let $S_1$ be a maximal separable subfield of $L_1$ of high codegree. Then $L_1/S_1$ is purely inseparable and $L_1$ has at most $\log_p[L_1 : K(L_1')]$ generators over $S_1$. But $\log_p[L_1 : K(L_1')] \leq \log_p[L : K(L^p)] = \text{some fixed constant}$ [8, Lemma 1, p. 111]. There must be an element, say $b$, of large exponent, say $n$, over $S_1$. Let $S$ be a maximal separable extension of $K$ in $L$ containing $S_1$. $S$ exists by Zorn’s Lemma. Then $S \cap L_1 = S_1$. Thus $b$ is of exponent $n$ over $S$, and hence $[L : S] \geq p^n$. Thus $L$ has maximal separable subfields of large codegree.

Corollary 6. If $\{(L : S) | S$ is a maximal separable extension of $K$ in $L\}$ is bounded, then any intermediate field $L_1$ over which $L$ is not algebraic must be separable over $K$.

Proof. Apply Theorem 4 and Lemma 5.

An intermediate field $L_1$ of $L/K$ has the same inseparability over $K$ as does $L$ if and only if $L^p$ and $K(L_1')$ are linearly disjoint over $L_1'$ [8, Lemma 1, p. 111]. The proof of [1, Theorem 1.4, p. 657] shows there is a unique minimal intermediate field $L_1$ which has $\text{insep}(L_1/K) = \text{insep}(L/K)$. ($L_1$ is merely the intersection of all subfields of $L_1$ with $\text{insep}(L_1/K) = \text{insep}(L/K)$.)

Theorem 7. Assume $\text{insep}(L/K) = 1$. The following are equivalent.

1) $\{(L : S) | S$ is maximal separable extension of $K$ in $L\}$ is bounded.
2) Any intermediate field $L_1$ over which $L$ is not algebraic must be separable over $K$.
3) $L$ is algebraic over $L_1$.

Proof. (1) implies (2) is Corollary 6. Assume (2). Since $L_1$ is inseparable over $K$, $L$ must be algebraic over $L_1$. Assume (3). Let $S$ be maximal separable and let $b \in L \setminus S$ with $b^p \in S$. Then $S(b)$ is inseparable over $K$, and hence $S(b) \supseteq L_1$. Thus $[L : S] \leq p \cdot \frac{L}{L_1}$.

Example 8. We give a family of extensions $L_n/K$ where $\text{inor}(L_n/K) = 1$ and the bound of Theorem 7 is exactly $p^{n+1}$. Let $L_n = K(x, z, uz^{p^n} + xv + w), K = P(u^p, v^p, w^p)$ where $P$ is a perfect field of characteristic $p \neq 0$ and $\{x, z, u, v, w\}$ is algebraically independent over $P$. $L_n$ has a subfield $L_{n_1} = K(x, z^{p^n}, uz^{p^n} + xv + w)$ which is separable algebraic over its irreducible form [2, Example 11, p. 190] and [2, Corollary 7, p. 188], call it $L_1$. Since $\text{inor}(L_{n_1}/K) = \text{inor}(L_n/K) = 1$, $L_1$ is the irreducible form of $L_n/K$. Clearly $[L_n : L_{n_1}] = p^n$. Let $S$ be a maximal separable extension of $K$ in $L_n$, and let $b \in L_n \setminus S$ with $b^p \in S$. Then $\text{inor}(S(b)/K) = 1$, and hence $S(b)$ must contain $L_1$. But $L_{n_1}$ is separable algebraic over $L_1$, and hence is contained in $S(b)$ since $L_n$ is purely inseparable over $S(b)$. Thus $[L_n : S(b)] \leq [L_n : L_{n_1}] = p^n$. Thus $[L_n : S] \leq p^{n+1}$. But $K(x, uz^{p^n} + xv + w)$ is a maximal separable extension of $K$ in $L_n$ (see the comments preceding Theorem 4) which is of codegree $p^{n+1}$. Thus the bound of Theorem 7 is precisely $p^{n+1}$.
**Theorem 9.** Assume \( \text{tr.d.}(L/K) = 1 \). The following are equivalent.

(1) \( \bar{K}/K \) is separable.

(2) There is an integer \( c \) such that any intermediate field \( L_1 \) over which \( L \) is purely inseparable and \( [L : L_1] > p^c \) must be separable over \( K \).

(3) \( \{[L : S] : S \text{ is maximal separable} \} \) is bounded.

**Proof.** Assume (1). The proof is by induction on \( \text{inor}(L/K) \). The result is trivially true for \( \text{inor}(L/K) = 0 \). Assume the result for \( \text{inor}(L/K) < n - 1 \) and let \( \text{inor}(L/K) = n \). Let \( L \) be purely inseparable over \( L_1 \) and suppose \( L_1 \) is inseparable over \( K \). We need to show \( [L : L_1] \) must be bounded for all such \( L_1 \). If \( L_1 \) contains a relatively \( p \)-independent element \( x \) of \( L/K \), then \( L_1 \) contains the separable algebraic closure of \( K(x) \), denoted \( (K(x))^\circ \), in \( L \), since \( L/L_1 \) is purely inseparable. By the comments preceding Theorem 4, \( (K(x))^\circ \) is a maximal separable extension of \( K \) in \( L \). By [7, Theorem 1, p. 353], the degrees of \( L \) over its maximal separable intermediate fields is bounded, and since \( L_1 \supseteq (K(x))^\circ \), the degree of \( L \) over \( L_1 \) is bounded. If \( L_1 \) does not contain a relatively \( p \)-independent element, then \( L_1 \subseteq K(L^p) \). By [1, Lemma 1.1, p. 656], \( \text{inor}(K(L^p)/K) < \text{inor}(L/K) \) when \( \text{inor}(L/K) > 0 \). Thus by induction, the degree of \( K(L^p) \) over \( L_1 \) is bounded, and hence also the degree of \( L \) over \( L_1 \). Clearly (2) implies (1) since \( [L : K(L^p)] > p^c \). [7, Theorem 1, p. 353] shows (1) is equivalent to (3).

**Theorem 10.** \( \{[L : S] : S \text{ is a maximal separable extension of } K \text{ in } L \} \) is bounded if and only if there is an integer \( c \) such that any intermediate field \( L_1 \) over which \( L \) is purely inseparable and \( [L : L_1] > p^c \) must be separable over \( K \).

**Proof.** If \( S \) is maximal separable and \( b \in L \setminus S \) with \( b^p \in S \), then \( S(b) \) is inseparable over \( K \). Thus, the existence of \( c \) guarantees \( [L : S(b)] \leq p^c \) and hence \( [L : S] \leq p^{c+1} \). Now assume there is a bound on the codegrees of maximal separable subfields. We prove there is a \( c \) by induction on the transcendence degree of \( L/K \).

The case of transcendence degree 1 is Theorem 9. We assume there is a sequence \( \{L_n\} \) of subfields of increasing codegree which are inseparable over \( K \), with \( L/L_n \) purely inseparable, and get a contradiction.

Let \( x \) be a relatively \( p \)-independent element of \( L/K \). Since the codegrees of maximal separable subfields is bounded, \( \bar{K} \) is separable over \( K \) [7, Corollary 2, p. 354]. Thus \( x \) is transcendental over \( K \). Since \( x \) is also relatively \( p \)-independent in \( L/K \), any maximal separable extension of \( K(x) \) in \( L \) is also a maximal separable extension of \( K \) in \( L \). Thus there is a bound on the codegrees of maximal separable extensions of \( K(x) \) in \( L \), and hence by induction, there is a \( c \) for \( L/K(x) \). Since \( x \) is transcendental over \( K \), each \( L_n(x) \) is inseparable over \( K(x) \). Thus the set of codegrees of the \( L_n(x) \) is bounded.

Let \( [L : L_n] = p^{d_n} \) where \( d_n \) is an increasing sequence. Let \( [L : L_n(x)] \leq p^{c_1} \) where \( c_1 \) is a constant. Consider the finite sequence \( a_1, a_2, \ldots, a_{d_n} \) defined by \( p^{d_n} = [L_n^{p^{d_n - x}}] \cap L \). Note that \( [L : L_n] = p^{a_1 + a_2 + \cdots + a_{d_n}} = p^{d_n} \). Since \( [L : L_n(x)] \leq p^{c_1} \), \( x \) is of exponent at least \( d_n - c_1 \) over \( L_n \). Thus, for \( i = 1, \ldots, d_n - c_1 \), \( a_i \geq 1 \), and at most the last \( c_1 \) of the \( a_i \)'s are 0. Since \( a_1 + \cdots + a_{d_n} = d_n \), at most \( c_1 \) of the \( a_i \)'s can exceed 1. So, we have a finite sequence of increasing length.
(d_n) with at most a fixed number of elements (2c_j) different from 1. Thus we can find strings of consecutive 1's of increasing length, say w_n, which begin at least as sequence element a_{d_n-w_n} for the sequence associated to L_n. Thus for s = \text{inor}(L/K) + 1, we can find, for large n, fields L'_n \supseteq L_n such that L'_n \cap L is simple over L'_n. Rename this sequence as \{L_n\}.

We now want to see that L'_n \cap L has the same order of inseparability over K as L'_n \cap L has over K, that is L'_n \cap L/K is a form of L_n \cap L/K. We can write L'_n \cap L = L_n(\theta) and L'_n \cap L = L_n(\theta^p). Now, the increase in the order of inseparability of L_n(\theta) depends upon \min\{\max\{r | \theta^r \in K(L'_n(\theta))\}, s\} \text{[4, Theorem 2, p. 374]. But this minimum must be } \max\{r | \theta^r \in K(L'_n(\theta))\} < s \text{ since } s > \text{inor}(L/K). \text{ Since the increase in the order of inseparability of } L'_n(\theta^p) = \min\{\max\{r | \theta^r \in K(L'_n(\theta))\}, s - 1\}, \text{ the increases will be the same, i.e., } L'_n \cap L/K \text{ is a form of } L_n \cap L/K.

By \text{[3, Theorem 3.3]}, L'_n \cap L/K has a distinguished subfield D_n not contained in any of L'_n \cap L. We claim D_n is a maximal separable subfield of L/K. Clearly D_n/K is separable and L/D_n is purely inseparable. Suppose x^p \in D_n, x \notin D_n. If x \in L'_n \cap L then x^p \in K(D'_n) since D_n is maximal separable in L'_n \cap L. If x \notin L'_n \cap L, then x must be in L'_n \cap L since L'_n \cap L is simple over L_n by construction. If x^p were not in K(D'_n), then by a degree argument D_n(x) would be distinguished for L'_n \cap L/K, a contradiction. Thus the sequence of \{D_n\} is a set of maximal separable extensions of K of unbounded codegree, a contradiction. Thus there is a c as in the statement of the theorem.

It is clear that the existence of a c as in the previous theorem implies that any subfield L_1 over which L is not algebraic must be separable over K. The converse is true in the transcendence degree (L/K) = 1 case, Theorem 9, or the insep(L/K) = 1 case, Theorem 7. We conjecture that it is true in general.

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