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AN ANALOGUE OF HILBERT'S THEOREM 90

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ABSTRACT. We prove a theorem on p-adic analytic functions that formally resembles Hilbert's Theorem 90 and state a general open problem of which it is a special case.

Let p be a prime, $\mathbf{Q}_p$ the field of p-adic numbers, $\mathbf{Z}_p$ the ring of p-adic integers, and $\mathbf{C}_p$ the completion of an algebraic closure of $\mathbf{Q}_p$. Let $|\ |$ denote the absolute value on $\mathbf{C}_p$. Fix $l \in \mathbf{Z}_p^\times$. We can define a function x^l for $|x-1|<1$ by the binomial series

$$x^l = ((x-1)+1)^l = \sum_{n=0}^{\infty} \binom{l}{n}(x-1)^n,$$

where $\binom{l}{n} = l(l-1)\cdot \cdots \cdot(l-n+1)/n!$. Suppose G is a nonvanishing analytic function defined for $|x-1|<1$ and put $F(x) = G(x)/G(x^l)$. Then F is analytic on $|x-1|<1$ and clearly satisfies

(*) If $x \in \mathbf{C}_p$, $f, n \in \mathbf{Z}_{\geqslant 0}$ are such that $x^{p^n} = 1$ and $l^f \equiv 1 \pmod{p^n}$, then $\prod_{j=0}^{f-1} F(x^{l^j}) = 1$.

Deligne has asked whether the converse of this is true: If

$$F(x) = \sum_{n=0}^{\infty} a_n(x-1)^n \in \mathbf{Z}_p[[x-1]]$$

satisfies (*), does there exist

$$G(x) = \sum_{n=0}^{\infty} b_n(x-1)^n \in \mathbf{Z}_p[[x-1]]$$

such that $F(x) = G(x)/G(x^l)$? We prove that this is the case if l is a root of unity.

REMARK. This result bears a strong formal similarity to theorems of Katz [2, Theorem 5] and myself [1], although the proofs of these three theorems are quite different. All three statements resemble Hilbert's Theorem 90, but I do not know of any unifying principle that explains such results.

THEOREM. *Suppose that either* $l^{p-1} = 1$ *or that* $p = 2$ *and* $l = -1$. *Let* $F(x) = \sum_{n=0}^{\infty} a_n(x-1)^n \in \mathbf{C}_p[[x-1]]$ *and assume* (i) $|a_n| \leqslant 1$ *for all* n *and* (ii) *the field*

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$K = \mathbf{Q}_p(a_0, a_1, a_2, \ldots)$ is discretely valued. If $(*)$ holds, then there exists $G(x) = \sum_{n=0}^{\infty} b_n(x-1)^n \in K[[x-1]]$, satisfying $|b_n| \leq 1$ for all n, such that $F(x) = G(x)/G(x^l)$.

PROOF. We consider first the case where $l^{p-1} = 1$. Put $H(x) = \prod_{j=0}^{p-2} F(x^{l^j})$. Taking $f = p-1$ in $(*)$ shows that $H(x) = 1$ for all x such that $x^{p^n} = 1$ for some n. The function $H(x) - 1$ thus has infinitely many zeros, but its series expansion in powers of $x - 1$ has coefficients in K. Hypotheses (i) and (ii) then imply, by an easy (and well-known) Newton polygon argument, that $H(x) - 1$ is identically zero:

$$\prod_{j=0}^{p-2} F(x^{l^j}) = 1 \quad \text{for all } x \text{ with } |x-1| < 1. \tag{1}$$

Let $\mathcal{O}_K$ be the ring of integers of K and consider the product $\prod_{j=0}^{p-3} F(x^{l^j})^{p-2-j}$. This lies in $\mathcal{O}_K[[x-1]]$ by hypothesis (i); actually, by $(*)$ with $x = 1$, $f = 1$, $n = 0$, we have $a_0 = 1$, so $J(x) \in 1 + (x-1)\mathcal{O}_K[[x-1]]$. Define $G(x)$ by the binomial series

$$G(x) = J(x)^{1/(p-1)} \in 1 + (x-1)\mathcal{O}_K[[x-1]].$$

Then

$$\frac{G(x)}{G(x^l)} = \left[\prod_{j=0}^{p-3} F(x^{l^j})^{p-2-j} \Big/ \prod_{j=0}^{p-3} F(x^{l^{j+1}})^{p-2-j}\right]^{1/(p-1)}$$

$$= \left[F(x)^{p-2} \Big/ \prod_{j=l}^{p-2} F(x^{l^j})\right]^{1/(p-1)}.$$

But $\prod_{j=1}^{p-2} F(x^{l^j}) = F(x)^{-1}$ by (1), hence $G(x)/G(x^l) = F(x)$.

Now suppose $p = 2$, $l = -1$. Taking $f = 2$ in $(*)$ shows that $F(x)F(1/x) = 1$ for all x such that $x^{p^n} = 1$ for some n. The same Newton polygon argument as above shows that in fact $F(x)F(1/x) = 1$ for all x with $|x-1| < 1$. Put $G(x) = 1 + F(x)$. Then

$$G(1/x) = 1 + F(x)^{-1},$$

from which it follows immediately that $G(x)/G(1/x) = F(x)$.

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