PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 1, May 1983

LENGTH AND AREA ESTIMATES OF THE DERIVATIVES OF BOUNDED HOLOMORPHIC FUNCTIONS

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ABSTRACT. MacGregor [1] and Yamashita [5] proved the estimates of the coefficient a_n of the Taylor expansion $f(z) = a_0 + a_n z^n + \cdots$ of f nonconstant and holomorphic in $|z| < 1$ in terms of the area of the image of $|z| < r < 1$ by f and the length of its outer or exact outer boundary. We shall consider some analogous estimates in terms of the non-Euclidean geometry for f bounded, $|f| < 1$, in $|z| < 1$. For example, $2\pi r^n |a_n|/(1 - |a_0|^2)$ is strictly less than the non-Euclidean length of the boundary of the image of $|z| < r$, the multiplicity not being counted.

1. Introduction. Unless otherwise specified, by f we always mean a function nonconstant, holomorphic, and bounded, $|f| < 1$, in the disk $U = \{|z| < 1\}$. The non-Euclidean metric in U is expressed by the differential form $\mu(z)|dz|$, $\mu(z) = (1 - |z|^2)^{-1}$, $z \in U$. Then $\Delta(z, r) = \{w \in U; |w - z|/|1 - \bar{z}w| < r\}$ is the non-Euclidean disk of the non-Euclidean center $z \in U$ and the non-Euclidean radius $(1/2)\log[(1 + r)/(1 - r)]$ $(0 < r < 1)$.

By the image $g(G)$ of a domain G by a function g holomorphic in G we mean the set of w in the w-plane such that $w = g(z)$ for at least one $z \in G$; simply, $g(G)$ is the projection of the Riemannian image of G by g. The exact outer boundary $C^\sharp(r, z)$ of $D(r, z) \equiv D(r, z, f) = f(\Delta(z, r))$ is the boundary of the unbounded component of the complement of the closure of $D(r, z)$ in the plane; see [5]. Roughly, $C^\sharp(r, z)$ is the boundary $\partial D(r, z)$ of $D(r, z)$ minus the "shorelines" of the "bays" and the "lakes" of the "island" $D(r, z)$. Furthermore, $C^\sharp(r, z)$ is a Jordan curve consisting of a finite number of analytic arcs. Let

$$\lambda(r, z) = \int_{C^\sharp(r, z)} \mu(w)\,|dw|$$

be the non-Euclidean length of $C^\sharp(r, z)$. The non-Euclidean length of $\partial D(r, z)$ is thus not smaller than $\lambda(r, z) > 0$.

A non-Euclidean version of S. Yamashita's estimate [5, Theorem 2] is

THEOREM 1. *Let f be nonconstant, holomorphic, and bounded, $|f| < 1$, in U. Let $n \equiv n(z)$ be the first number such that $f^{(n)}(z) \neq 0$, $n \geq 1$, $z \in U$. Then, for each r, $0 < r < 1$,*

$$(1.1) \qquad 2\pi r^n (1 - |z|^2)^n |f^{(n)}(z)| / \left[n!\,(1 - |f(z)|^2)\right] \leq \Phi(\lambda(r, z)) < \lambda(r, z),$$

where, for $0 \leq x < +\infty$, $\Phi(x) \geq 0$ and $\Phi(x)^2 = 2\pi(\pi^2 + x^2)^{1/2} - 2\pi^2$.

Received by the editors March 24, 1982
1980 *Mathematics Subject Classification.* Primary 30C99; Secondary 30C50, 30C80.

We note that $(0 <)\ \Phi(x) < x$ for $x > 0$, so that the second inequality in (1.1) is immediate.

In particular, (1.1) for $z = 0$, together with

$$f(z) = a_0 + a_n z^n + \cdots \qquad (a_n \neq 0), \tag{1.2}$$

yields that

$$2\pi r^n |a_n| / (1 - |a_0|^2) \leqslant \Phi(\lambda(r,0)) < \lambda(r,0);$$

this is a non-Euclidean counterpart of [**5**, Theorem 1].

For the proof of Theorem 1 we shall make use of the following Theorem 2; unfortunately, the formulation appears to be complicated.

Let E be the unbounded component of the complement of $D(r, z)$, not that of the closure of $D(r, z)$. Let $\hat{D}(r, z)$ be the complement of E. Then $\hat{D}(r, z)$ consists of the "island" $D(r, z)$ plus its reclaimed "lakes". As is pointed out by T. H. MacGregor [**1**, p. 319; **2**, Lemma 2], the domain $\hat{D}(r, z)$ is simply connected whose boundary is called the outer boundary of $D(r, z)$. Thus, for $f(z) = (z + 1/\sqrt{3})^3/8$, $D(r,0)$ is a proper subset of $\hat{D}(r,0)$ if $1/2 < r < 1$, while $D(r,0) = \hat{D}(r,0)$ if $0 < r \leqslant 1/2$.

Returning to our general f we let

$$\alpha(r, z) \equiv \alpha(r, z, f) = \iint_{\hat{D}(r,z)} \mu(w)^2\, dx\, dy \qquad (w = x + iy)$$

be the non-Euclidean area of $\hat{D}(r, z)$ $(0 < r < 1,\ z \in U)$.

THEOREM 2. *Let f and $n(z)$ be as in Theorem* 1. *Then, for each r, $0 < r < 1$,*

$$\pi r^{2n}(1 - |z|^2)^{2n} |f^{(n)}(z)/[n!\,(1 - |f(z)|^2)]|^2 \leqslant \alpha(r, z). \tag{1.3}$$

Specifically, (1.3) for $z = 0$ with (1.2) reads

$$\pi r^{2n}[|a_n|/(1 - |a_0|^2)]^2 \leqslant \alpha(r,0), \tag{1.4}$$

for all r, $0 < r < 1$. This is a non-Euclidean counterpart of MacGregor's estimate [**1**, Theorem 1]:

$$\pi r^{2n} |a_n|^2 \leqslant a(r,0), \tag{1.5}$$

where $a(r,0)$ is the Euclidean area of $D(r,0)$, not that of $\hat{D}(r,0)$ for f of (1.2) not necessarily bounded in U. Since $a(r,0) \leqslant \alpha(r,0)$ for $|f| < 1$, (1.4) yields a better estimate than (1.5) if $|a_0|$ is so near 1 that

$$(1 - |a_0|^2)^2 \alpha(r,0) \leqslant a(r,0).$$

2. Proofs. In the proof of his theorem [**1**, Theorem 1] MacGregor makes use of the following improvement of [**3**, Theorem 4.7, p. 80].

MACGREGOR'S LEMMA. *Let*

$$g(z) = b_0 + b_n z^n + \cdots \qquad (b_n \neq 0,\ n \geqslant 1)$$

be holomorphic in U, and let r_1 be the inner radius [**3**, *p.* 79] *of $g(U)$ at b_0. Then*

$$|b_n| \leqslant r_1. \tag{2.1}$$

PROOF OF THEOREM 2. First we prove (1.4), then (1.3). For the proof of (1.4) we let

$$g(z) \equiv f(rz) = a_0 + a_n r^n z^n + \cdots \quad \text{in } U,$$

and let r_1 be the inner radius of $g(U) = D(r,0)$ at a_0. Then the estimate (2.1), together with $b_n = a_n r^n$, yields

$$|a_n| r^n \leqslant r_1. \tag{2.2}$$

Let r_2 be the inner radius of $D\hat{\ }(r,0)$ at a_0. Then $r_1 \leqslant r_2$ because $D(r,0) \subset D\hat{\ }(r,0)$; see [**3**, p. 80].

Let D^* be the circular symmetrization [**3**, p. 69] of $D\hat{\ }(r,0)$ with respect to the half-line $\{ta_0; 0 \leqslant t < +\infty\}$ ($= \{t; 0 \leqslant t < +\infty\}$, if $a_0 = 0$). Then D^* is simply connected because the same is true of $D\hat{\ }(r,0)$. Let $h(z) = a_0 + c_1 z + \cdots$ be a univalent holomorphic function in U with $h(U) = D^*$. Then the inner radius r_3 of D^* at a_0 satisfies $r_3 = |c_1|$ [**3**, p. 79], and by [**3**, Theorem 4.8, p. 81], $r_2 \leqslant r_3$, so that, by (2.2),

$$|a_n| r^n \leqslant r_3 = |c_1| = |h'(0)|,$$

whence follows

$$r^{2n}\left[|a_n|/\left(1 - |a_0|^2\right)\right]^2 \leqslant |h'(0)|^2/\left(1 - |h(0)|^2\right)^2, \tag{2.3}$$

because $h(0) = a_0$.

Since $|h'|^2/(1 - |h|^2)^2$ is subharmonic in U, and since the non-Euclidean area of D^* is the same as that of $D\hat{\ }(r,0)$, or, $\alpha(r,0)$, it follows that

$$\begin{aligned} |h'(0)|^2/\left(1 - |h(0)|^2\right)^2 &\leqslant (1/\pi)\iint_U |h'(z)|^2/\left(1 - |h(z)|^2\right)^2 dx\, dy \\ &= \alpha(r,0)/\pi, \end{aligned}$$

which, together with (2.3), yields (1.4).

To prove (1.3) we consider the composed function

$$F(w) = f((w + z)/(1 + \bar{z}w))$$

of a variable $w \in U$. Since

$$F^{(n)}(0)/n! = \left(1 - |z|^2\right)^n f^{(n)}(z)/n!,$$

$F(0) = f(z)$, and since $\alpha(r, 0, F) = \alpha(r, z, f)$, (1.3) is a consequence of (1.4) applied to F.

PROOF OF THEOREM 1. The Gauss curvature $K(z)$ of the non-Euclidean space U endowed with the metric $\mu(z)|dz|$ is given by

$$K(z) = -4\mu(z)^{-2}(\partial^2/\partial z\, \partial \bar{z})\log \mu(z) \equiv -4 \quad \text{in } U.$$

Let $D^\sharp(r, z)$ be the domain bounded by the Jordan curve $C^\sharp(r, z)$. Then $D(r, z) \subset D\hat{\ }(r, z) \subset D^\sharp(r, z)$. Let A be the non-Euclidean area of $D^\sharp(r, z)$. Then

$$\alpha(r, z) \leqslant A \quad \text{and} \quad 4\pi A + 4A^2 \leqslant \lambda(r, z)^2;$$

the latter is a consequence of [**4**, (4.25), p. 1206], together with $K \equiv -4$. Consequently,

$$2\alpha(r, z) \leqslant (\pi^2 + \lambda^2)^{1/2} - \pi.$$

The estimate (1.1) now follows from (1.3) after a short computation.

3. Schwarz-Pick's lemma. As applications of Theorems 1 and 2 we obtain improvements of Schwarz-Pick's lemma:

$$(1 - |z|^2)|f'(z)|/(1 - |f(z)|^2) \leqslant 1, \qquad z \in U.$$

For example, let

$$M(r, z) = \min(1, \Phi(\lambda(r, z))/(2\pi r)),$$

for $0 < r < 1$, $z \in U$. If $f'(z) \neq 0$, then we obtain by (1.1) that

$$(1 - |z|^2)|f'(z)|/(1 - |f(z)|^2) \leqslant M(r, z),$$

while if $f'(z) = 0$, then the estimate is trivial. The estimate in terms of $\alpha(r, z)$ is similar, and is left as an exercise.

4. Concluding remarks. As to the sharpness of (1.4) on which (1.3) depends we have poor information: (1.4) *is sharp in the limiting case*, $r \to 0$. More precisely, let us be given $n \geqslant 1$ and $a_0 \in D$. We set

$$T(z) = (z + a_0)/(1 + \overline{a_0} z) = a_0 + bz + \cdots,$$

and

$$f(z) = T(z^n) = a_0 + a_n z^n + \cdots,$$

where $a_n = b$. Then, $\alpha(r, 0) = \alpha(r, 0, f)$ is the non-Euclidean area of $\Delta(a_0, r^n)$ which is the same as that of $\Delta(0, r^n)$, or $\alpha(r, 0) = \pi r^{2n}/(1 - r^{2n})$. Since

$$|a_n|/(1 - |a_0|^2) = |b|/(1 - |a_0|^2) = 1,$$

(1.4) now reads

$$\pi r^{2n} = \pi r^{2n}\left[|a_n|/(1 - |a_0|^2)\right]^2 \leqslant \pi r^{2n}/(1 - r^{2n}),$$

whence, the fact that $1 - r^{2n} \to 1$ as $r \to 0$ yields the sharpness in the limit.

Now, the situation explained at the end of §1 is obvious for the present f. For, given $0 < r < 1$, we choose a real a_0 so that

$$0 < a_0 < 1 \quad \text{and} \quad (1 - a_0^2 r^{2n})^2 < 1 - r^{2n}.$$

A calculation shows that

$$a(r, 0) = \pi r^{2n}(1 - a_0^2)^2/(1 - a_0^2 r^{2n})^2,$$

so that $(1 - a_0^2)^2 \alpha(r, 0) < a(r, 0)$.

Conversely, given a complex number a_0 with $1/\sqrt{2} < |a_0| < 1$, then for each r with

$$0 < r < |a_0| < 1 \quad \text{and} \quad (1 - |a_0|^2 r^{2n})^2 < 1 - r^{2n},$$

the same argument as above again shows that

$$\left(1-|a_0|^2\right)^2\alpha(r,0)<a(r,0).$$

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