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ON A TAUBERIAN THEOREM FOR THE L^1-CONVERGENCE OF FOURIER SINE SERIES

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ABSTRACT. In a recent Tauberian theorem of Stanojević [3] for the L^1-convergence of Fourier series, the notion of asymptotically even sequences is introduced. These conditions are satisfied if the Fourier coefficients $\{\hat{f}(n)\}$ are even ($\hat{f}(-n) = \hat{f}(n)$), a case formally equivalent to cosine Fourier series. This paper applies the Tauberian method of Stanojević [3] separately to cosine and sine Fourier series and shows that the notion of asymptotic evenness can be circumvented in each case.

Classically it is well known that there exists an $f \in L^1(-\pi, \pi)$ whose Fourier series

(1) $$S(f) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

diverges in $L^1(-\pi, \pi)$-norm as n tends to infinity. That is to say $\| S_n(f) - f \| \neq o(1)$ $(n \to \infty)$, where

(2) $$S_n(f) = S_n(f, x) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx)$$

are the partial sums of (1) and,

$$\| f \| = \frac{1}{2\pi} \int_{-\pi}^{\pi} | f(x) | \, dx, \qquad f \in L^1(-\pi, \pi).$$

However, in the case of cosine Fourier series

(C) $$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx,$$

many authors, through the use of regularity and/or speed conditions on the sequence $\{a_n\}_0^\infty$, have defined subclasses of $L^1(0, \pi]$ for which

(3) $\| S_n(f) - f \| = o(1)$ $(n \to \infty)$ if and only if $a_n \lg n = o(1)$ $(n \to \infty)$.

For example, if $a_n = o(1)$ $(n \to \infty)$ and $\sum_{n=1}^{\infty} (n+1) | \Delta^2 a_n |$ converges, then Kolmogorov [1] proved that C is the Fourier series of some $f \in L^1(0, \pi]$ and (3) holds (see [2] for a further survey). Analogous results have been obtained for sine series

$(\tilde{C})$ $$\sum_{n=1}^{\infty} b_n \sin nx,$$

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(3) being replaced by

(4) $\|\tilde{S}_n(f) - f\| = o(1)$ $(n \to \infty)$ if and only if $b_n \lg n = o(1)$ $(n \to \infty)$

where

$$\tilde{S}_n(f) = \tilde{S}_n(f, x) = \sum_{k=1}^{n} b_k \sin kx.$$

A recent result on this problem is the following Tauberian theorem [3] for the L^1-convergence of Fourier series of complex valued Lebesque integrable functions on $T = \mathbf{R}/2\pi Z$ (the class of such functions is denoted $L^1(T)$). Throughout this paper, $\Delta c(k) = c(k) - c(k+1)$.

THEOREM (STANOJEVIĆ [3]). *Let*

$$S(f) \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}$$

be the Fourier series of $f \in L^1(T)$ *whose coefficients satisfy*

$$\frac{1}{n} \sum_{k=1}^{n} |\hat{f}(k) - \hat{f}(-k)| \lg k = o(1) \qquad (n \to \infty), \tag{5}$$

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{k=n}^{[\lambda n]} |\Delta(\hat{f}(k) - \hat{f}(-k))| \lg k = 0. \tag{6}$$

If for some $1 < p \leq 2$,

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta \hat{f}(k)|^p = 0, \tag{7}$$

then $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ *if and only if*

$$\|\hat{f}(n) E_n + \hat{f}(-n) E_{-n}\| = o(1) \qquad (n \to \infty), \tag{8}$$

where

$$E_m = E_m(t) = \sum_{k=0}^{m} e^{ikt}, \qquad m \in Z.$$

Sequences satisfying conditions (5) and (6) are called asymptotically even due to the fact that the difference $\hat{f}(n) - \hat{f}(-n)$ is a constant multiple of the nth-sine coefficient of f. Also note that in the cases of even coefficients ($\hat{f}(n) = \hat{f}(-n)$) or odd coefficients ($\hat{f}(n) = -\hat{f}(-n)$), condition (8) is equivalent with $\hat{f}(n) \lg n = o(1)$ $(n \to \infty)$. The Tauberian nature of this result is most transparent in the following corollary [3].

COROLLARY. *Let* $S(f)$ *be as in Stanojević's theorem. If* $n\Delta\hat{f}(n) = O(1)$ $(n \to \infty)$ *then* $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ *if and only if* (8).

The purpose of the present paper is to extend Stanojević's theorem to sine Fourier series generally without the notion of asymptotic evenness. More precisely, the following theorem is proven.

THEOREM 1. *Let $\tilde{C}$ be the Fourier series of $f \in L^1(0, \pi]$. If for some $1 < p \leqslant 2$,*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta b_k|^p = 0, \tag{9}$$

then (4) *holds.*

PROOF. Denote the partial sums of $\tilde{C}$ by $\tilde{S}_n(f, x)$ and consider for $\lambda > 1$, the truncated Cesàro means,

$$V_n^\lambda(f, x) = \frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} \tilde{S}_k(f, x).$$

Then denoting the usual $(C, 1)$ means by $\sigma_n(f, x)$ we have

$$V_n^\lambda(f, x) = \frac{[\lambda n] + 1}{[\lambda n] - n} \sigma_{[\lambda n]}(f, x) - \frac{n+1}{[\lambda n] - n} \sigma_n(f, x).$$

Since $\|\sigma_n(f, x) - f\| = o(1)$ $(n \to \infty)$, it then follows that $\|V_n^\lambda(f) - f\| = o(1)$ $(n \to \infty)$. Consequently, it suffices to prove that

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \|V_n^\lambda(f) - \tilde{S}_n(f)\| = 0 \tag{10}$$

if and only if $b_n \lg n = o(1)$ $(n \to \infty)$. Elementary calculation gives the identity

$$V_n^\lambda(f, x) - \tilde{S}_n(f, x) = \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k + 1}{[\lambda n] - n} b_k \sin kx, \tag{11}$$

and the norm of this difference is estimated through the following device,

$$\|V_n^\lambda(f) - \tilde{S}_n(f)\| = \frac{1}{\pi}\left\{\int_0^{\pi/n} + \int_{\pi/n}^{\pi}\right\} |V_n^\lambda(f, x) - \tilde{S}_n(f, x)|\, dx = I_1 + I_2.$$

The first integral is majorized by

$$I_1 \leqslant \frac{1}{n} \sum_{k=n+1}^{[\lambda n]} \frac{[\lambda n] - k + 1}{[\lambda n] - n} |b_k| \leqslant \frac{1}{n} \sum_{k=n+1}^{[\lambda n]} |b_k| = o(1) \qquad (n \to \infty),$$

since $b_n = o(1)$ $(n \to \infty)$. Consequently, (10) holds if and only if

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \int_{\pi/n}^{\pi} |V_n^\lambda(f, x) - \tilde{S}_n(f, x)|\, dx = 0.$$

To estimate I_2, summation by parts is applied to the right-hand side of (11), i.e.,

$$\begin{aligned} V_n^\lambda(f, x) - \tilde{S}_n(f, x) = &\sum_{k=n}^{[\lambda n]-1} \Delta\left(\frac{[\lambda n] - k + 1}{[\lambda n] - n} b_k\right) \tilde{D}_k(x) \\ &+ \frac{b_{[\lambda n]} \tilde{D}_{[\lambda n]}(x)}{[\lambda n] - n} - \frac{[\lambda n] - n + 1}{[\lambda n] - n} b_n \tilde{D}_n(x), \end{aligned} \tag{12}$$

where

$$\tilde{D}_n(x) = \sum_{k=1}^{n} \sin kx = \frac{\cos x/2 - \cos(n + 1/2)x}{2 \sin x/2}$$

is the conjugate Dirichlet kernel. Introducing a modified conjugate kernel $\bar{D}_n(x) = \tilde{D}_n(x) - \frac{1}{2}\cot x/2$, eliminating the collapsing sums in (12), and writing

$$\Delta(([\lambda n] - k + 1)b_k) = ([\lambda n] - k)\Delta b_k + b_k,$$

we get

$$V_n^\lambda(f, x) - \tilde{S}_n(f, x) = \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta b_k \bar{D}_k(x) + \sum_{k=n+1}^{[\lambda n]} \frac{b_k}{[\lambda n] - n} \bar{D}_k(x) - b_n \bar{D}_n(x).$$

Rearranging this, integrating over $(\pi/n, \pi]$ and applying the triangle inequalities yields the estimate

$$(13) \quad \left|\pi I_2 - |b_n| \int_{\pi/n}^{\pi} |\bar{D}_n(x)|\,dx\right| \leqslant \int_{\pi/n}^{\pi} \left| \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta b_k \bar{D}_k(x) \right| dx + \frac{1}{[\lambda n] - n} \int_{\pi/n}^{\pi} \left| \sum_{k=n+1}^{[\lambda n]} b_k \bar{D}_k(x) \right| dx = \Sigma_1 + \Sigma_2.$$

For the first sum on the right-hand side, the inequalities of Hölder and Hausdorff-Young are applied as follows ($1/p + 1/q = 1$):

$$\Sigma_1 \leqslant \left\{ \int_{\pi/n}^{\pi} \frac{dx}{(2 \sin x/2)^p} \right\}^{1/p} \left\{ \int_0^{\pi} \left| \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n] - k}{[\lambda n] - n} \Delta b_k \cos(k + \tfrac{1}{2})x \right|^q dx \right\}^{1/q}$$

$$\leqslant A_p n^{1/q} \left(\sum_{k=n}^{[\lambda n]-1} \left(\frac{[\lambda n] - k}{[\lambda n] - n} \right)^p |\Delta b_k|^p \right)^{1/p} \leqslant A_p \left(\sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta b_k|^p \right)^{1/p},$$

where A_p is a constant dependent on p only. The second term, Σ_2, is estimated in the same fashion to yield

$$\Sigma_2 \leqslant A_p \left(\frac{n}{[\lambda n] - n} \right)^{1/q} \left(\frac{1}{[\lambda n] - n} \sum_{k=n+1}^{[\lambda n]} |b_k|^p \right)^{1/p}.$$

Again since $b_n = o(1)$ ($n \to \infty$), we see that $\Sigma_2 = o(1)$ ($n \to \infty$) and thus, returning to (13),

$$\left| \lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \frac{\pi}{2} \int_{\pi/n}^{\pi} |V_n^\lambda(f, x) - \tilde{S}_n(f, x)|\,dx - \overline{\lim_{n \to \infty}} |b_n| \int_{\pi/n}^{\pi} |\bar{D}_n(x)|\,dx \right|$$

$$\leqslant A_p \lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \left(\sum_{k=n}^{[\lambda n]} k^{p-1} |\Delta b_k|^p \right)^{1/p} = 0.$$

The proof is concluded recalling the well-known fact that

$$A \lg n \leqslant \int_{\pi/n}^{\pi} |\bar{D}_n(x)|\,dx \leqslant B \lg n,$$

where A and B are absolute constants.

Theorem 1 has the following corollary that again reveals the Tauberian nature of the result.

COROLLARY 1. *Let* $\tilde{C}$ *be the Fourier series of* $f \in L^1(0, \pi)$. *If* $n\Delta b_n = O(1)$ $(n \to \infty)$ *then* (4).

PROOF. It suffices to show our hypothesis implies (9). Let M be such that $n \mid \Delta b_n \mid \leq M$, for all n. Then,

$$\sum_{k=n}^{[\lambda n]} k^{p-1} \mid \Delta b_k \mid^p \leq M^p \sum_{k=n}^{[\lambda n]} \frac{1}{k} = O(\lg \lambda)$$

from which the result follows.

In closing, note that Theorem 1, its corollary and proof hold in a similar fashion for cosine Fourier series.

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