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APPROXIMATE IDENTITIES AND $H^1(\mathbf{R})$

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ABSTRACT. Let $\varphi(x) \in L^1(\mathbf{R}) \cap L^\infty(\mathbf{R})$ be a real-valued function with $\int_{\mathbf{R}} \varphi\, dx \neq 0$. For $y > 0$, let $\varphi_y(x) = y^{-1}\varphi(x/y)$. For $f(x) \in L^1(\mathbf{R})$ define

$$f_\varphi^*(x) = \sup_{y>0, t\in\mathbf{R}: |x-t|<y} |f * \varphi_y(t)| .$$

We investigate the space $H_\varphi^1 = \{f \in L^1(\mathbf{R}): f_\varphi^* \in L^1(\mathbf{R})\}$.

1. Introduction. If φ is the Poisson kernel, then H_φ^1 is defined to be H^1. Fefferman and Stein [2] showed that $H_\varphi^1 = H^1$ for any φ that is smooth and dies quickly at infinity; e.g. φ can be in the Schwartz class, or Lipschitz continuous (of any order) and compactly supported. However, it is easy to show that $H_\varphi^1 = \{0\}$ if $\varphi = \chi_{[0,1]}$ (see [3]), where χ_E is the characteristic function of a set E. G. Weiss asked whether there was an H_φ^1 that was nontrivial but not H^1. In this note, we show the following two results.

THEOREM 1. *If $H_\varphi^1 \neq \{0\}$, then $a(x) \in H_\varphi^1$, where*

$$a(x) = \begin{cases} 1 & 0 < x < 1, \\ -1 & -1 < x < 0, \\ 0 & \textit{otherwise}. \end{cases}$$

THEOREM 2. *There exists $\varphi(x) \geqslant 0$ such that $H_\varphi^1 \neq \{0\}$, $H_\varphi^1 \neq H^1$.*

As a corollary of Theorem 1, we get

COROLLARY 1. *If $H_\varphi^1 \neq \{0\}$, then $H_\varphi^1 \cap H^1$ is dense in H^1.*

Comment on notation. To distinguish the "y" in $\varphi_y(x)$ $(= y^{-1}\varphi(x/y))$ from the other subindices, in the following we write $(\varphi)_y$ instead of φ_y. The letter C denotes various constants.

2. Proof of Theorem 1. For $f \in H_\varphi^1$ define

$$\| f \|_{H_\varphi^1} = \| f_\varphi^* \|_{L^1}.$$

This norm makes H_φ^1 a Banach space. We use two simple facts about $\| \; \|_{H_\varphi^1}$.

FACT 1. If $f \in H_\varphi^1$, $g \in L^1$, then $f * g \in H_\varphi^1$ with

$$\| f * g \|_{H_\varphi^1} \leqslant \| f \|_{H_\varphi^1} \| g \|_{L^1}.$$

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FACT 2. If $y > 0$ and $f \in H^1_\varphi$, then

$$\|(f)_y\|_{H^1_\varphi} = \|f\|_{H^1_\varphi}.$$

Let $f \in H^1_\varphi$, $f \not\equiv 0$ and fix f. In the following part of this section, the constants c depend on this function f. We may assume that f is real-valued (since φ is real-valued). We shall construct functions p_n, g_n $(-\infty < n < \infty)$ satisfying

$$\|p_n\|_{H^1_\varphi} \leqslant c, \tag{1}$$

$$\sum_{n=-\infty}^{\infty} \|g_n\|_{L^1} < +\infty, \tag{2}$$

$$a = \sum_{n=-\infty}^{\infty} p_n * g_n, \tag{3}$$

where the convergence is in H^1_φ. This implies the theorem.

The construction of p_n and g_n. Since $f \not\equiv 0$ and since f is real-valued, we may assume there exist $r > 1$ and $\varepsilon > 0$ such that

$$|\hat{f}(\xi)| > \varepsilon \quad \text{on } [-r, -r^{-1}] \cup [r^{-1}, r].$$

Let $\psi(x) \in \mathcal{S}(\mathbf{R})$ be a real-valued even function such that

$$\operatorname{supp} \hat{\psi} \subset [-r, -r^{-1}] \cup [r^{-1}, r], \qquad \sum_{k=-\infty}^{\infty} \hat{\psi}(r^k\xi)^2 \equiv 1 \quad \text{for any } \xi \neq 0.$$

Now we invoke Wiener's Lemma: Let $f_1(x), f_2(x) \in L^1(\mathbf{R})$. If there exist an $\varepsilon > 0$ and an interval $I \subset \mathbf{R}$ for which $|\hat{f}_1(\xi)| > \varepsilon$, $\xi \in I$, and $\operatorname{supp} \hat{f}_2 \subset I$, then there is an $h(x) \in L^1(\mathbf{R})$ such that $\hat{f}_2(\xi) = \hat{h}(\xi)\hat{f}_1(\xi)$.

Applying Wiener's Lemma to $f(x)$ and $(\hat{\psi}\chi_{(0,\infty)})^\vee$, we get $h_1(x) \in L^1(\mathbf{R})$ such that

$$\hat{\psi}(\xi)\chi_{(0,\infty)}(\xi) = \hat{h}_1(\xi)\hat{f}(\xi).$$

Set $\hat{h}(\xi) = \hat{h}_1(\xi) + \bar{\hat{h}}_1(-\xi)$. Then $\hat{\psi}(\xi) = \hat{h}(\xi)\hat{f}(\xi)$, and

$$\|\psi\|_{H^1_\varphi} \leqslant \|h\|_{L^1}\|f\|_{H^1_\varphi} \leqslant c\|f\|_{H^1_\varphi}. \tag{4}$$

We now define

$$p_n(x) = (\psi)_{r^n}(x), \quad g_n(x) = a * (\psi)_{r^n}(x).$$

Then (1) follows from (4). By taking Fourier transforms, we see that $a = \sum_{n=-\infty}^{\infty} p_n * g_n$ in $\mathcal{S}'$. To estimate $\|g_n\|_{L^1}$, we divide into two cases.

Case 1. $n \geqslant 0$. We write

$$\begin{aligned} |g_n(x)| &= \left|\int_{-1}^{1} r^{-n}\psi(r^{-n}(x-t))a(t)\,dt\right| \\ &= r^{-n}\left|\int_{-1}^{1} (\psi(r^{-n}(x-t)) - \psi(r^{-n}x))a(t)\,dt\right| \\ &\leqslant cr^{-2n} \sup_{|r^{-n}x - t| < r^{-n}} |\psi'(t)| \leqslant cr^{-2n}R(r^{-n}x), \end{aligned}$$

where $R(x) = \sup_{|x-y|<1} |\psi'(y)|$. Therefore,

$$\|g_n\|_{L^1} \leqslant cr^{-2n} \int R(r^{-n}x)\,dx \leqslant cr^{-n}.$$

Case 2. $n < 0$. We distinguish three subcases.

Subcase 1. $|x| > 3$.

$$|g_n(x)| = \left| \int_{-1}^{1} r^{-n}\psi(r^{-n}(x-t))a(t)\,dt \right| \leqslant cr^{-n}/(r^{-n}|x|)^4$$

(ψ is rapidly decreasing). Thus, $\int_{|x|>3} |g_n(x)| \leqslant cr^{3n}$.

Subcase 2. $|x| \leqslant 3$, $\min(|x|, |x+1|, |x-1|) \geqslant r^{n/2}$. These x's are away from the discontinuities of $a(x)$. We have

$$|g_n(x)| \leqslant \left| \int_{|x-t|<r^{n/2}} r^{-n}\psi(r^{-n}(x-t))a(t)\,dt \right| + \left| \int_{|x-t|>r^{n/2}} \cdots\,dt \right|$$

The second term can be estimated as in the first subcase. The first term equals zero or it equals $|\int_{|t|>r^{-n/2}} \psi(t)\,dt|$ (because $\int \psi(t)\,dt = 0$). This is dominated by cr^n, since ψ is rapidly decreasing.

Subcase 3. $\min(|x|, |x+1|, |x-1|) < r^{n/2}$. Here the best we can do is $|a * \psi_n(x)| \leqslant c$. But the measure of this set is $\leqslant 6r^{n/2}$.

Combining the three subcases yields for $n < 0$, $\|g_n\|_{L^1} \leqslant cr^{n/2}$. We therefore have (2).

3. Proof of Corollary 1. It is well known that the dual space of H^1 is the space BMO (see **[2]**). This is the space of locally integrable functions $h(x)$ that satisfy

$$\sup_I |I|^{-1} \int_I |h(x) - h_I|\,dx = \|h\|_* < \infty.$$

The supremum is over all intervals $I \subset \mathbf{R}$; h_I denotes the average of $h(x)$ over I.

Clearly $a(x) \in H^1$. Also H^1 and H^1_φ are closed under translations and dilations. If $H^1_\varphi \cap H^1$ is not dense, then there is an $h \in \text{BMO}$ such that $\|h\|_* = 1$ but $\int h(x)g(x)\,dx = 0$, for any $g \in H^1_\varphi \cap H^1$. The same must hold for any dilation or translation of $a(x)$. This implies that h is constant and $\|h\|_* = 0$.

4. Proof of Theorem 2. An examination of the proof of Theorem 1 shows that it works because of the relative smoothness of $a(x)$. In this section, we exhibit an H^1_φ that is not trivial or H^1, by building functions $b(x) \in H^1$ and $\varphi(x)$, each of which has "large" high frequency terms in its Fourier series. The high frequencies of $\varphi(x)$ almost cancel out when $\varphi(x)$ is convolved with $a(x)$, but they match up with those of $b(x)$ to make $b(x) \notin H^1_\varphi$.

For $n = 1, 2, 3, \ldots,$ define

$$\mu_n(x) = \sum_{k=1}^{n} \sin(2^k \pi x)\chi_{[1,2]}(x).$$

We estimate $|a * (\mu_n)_y(x)|$ as follows.

Case 1. $y < 1$.

$$|a * (\mu_n)_y(x)| \leqslant C \sum_{k=1}^{n} (1/y)(y/2^k) \leqslant C.$$

Case 2. $y > 2^n$.

$$|a * (\mu_n)_y(x)| \leqslant C \sum_{k=1}^{n} (1/y)(2^k/y) \leqslant C2^n/y^2.$$

Case 3. $1 \leqslant y \leqslant 2^n$.

$$|a * (\mu_n)_y(x)| \leqslant C \sum_{\log_2 y < k \leqslant n} (1/y)(y/2^k) + C \sum_{1 \leqslant k \leqslant \log_2 y} (1/y)(2^k/y) \leqslant C/y.$$

Now observe that $a * (\mu_n)_y(t) = 0$ if $y \leqslant (t-1)/2$ or $y \leqslant (-t-1)/2$. Thus

$$a^*_{\mu_n}(x) \leqslant \begin{cases} C & \text{if } |x| \leqslant 1, \\ C/|x| & \text{if } 1 \leqslant |x| \leqslant 6 \cdot 2^n, \\ C2^n/|x|^2 & \text{if } 6 \cdot 2^n \leqslant |x| . \end{cases}$$

This yields $\|a^*_{\mu_n}\|_{L^1} \leqslant Cn$.

If $\alpha > 1$, then by

$$a * (\mu_n(\alpha \cdot))_y(t) = \alpha^{-1} a * (\mu_n)_{y/\alpha}(t),$$

and by similar observations as above, we get

$$\|a^*_{\mu_n(\alpha\,\cdot)}\|_{L^1} \leqslant Cn, \tag{5}$$

where C does not depend on $\alpha > 1$.

Define

$$\alpha_n = 2^{2^n}, \qquad \eta(x) = \sum_{n=1}^{\infty} n^{-2-\varepsilon_0} \mu_n(\alpha_n x),$$

where $\varepsilon_0 > 0$ is a small number. Then, by (5) we have

$$\|a^*_\eta\|_{L^1} \leqslant \sum_n n^{-2-\varepsilon_0} \|a^*_{\mu_n(\alpha_n\,\cdot)}\|_{L^1} \leqslant C \sum_n n^{-1-\varepsilon_0} < +\infty. \tag{6}$$

Let $\varepsilon > 0$ be a small number. Define

$$b(x) = -\sum_{k=1}^{\infty} k^{-1+\varepsilon} \sin(2^k \pi x) \chi_{[-2,-1]}(x).$$

From the fact that $b \in L^2$, $\int b\, dx = 0$ and $\operatorname{supp} b \subset [-2,-1]$, it follows that $b \in H^1$ (see **[1]**).

We claim that for $n > N_\varepsilon$ and $0 \leqslant i \leqslant n/2$,

$$\left| \int b(x) \mu_n(2^{-i}(-x-1)+1)\, dx \right| \geqslant C_\varepsilon n^\varepsilon.$$

This is because the left-hand side equals

$$\left| \int b(x) \sum_{k=i+1}^{n} \sin\left(2^k\pi\left(2^{-i}(-x-1)+1\right)\right) dx \right.$$
$$\left. + \int b(x) \sum_{k=1}^{i} \sin\left(2^k\pi\left(2^{-i}(-x-1)+1\right)\right) dx \right|.$$

The first integral equals

$$\frac{1}{2} \sum_{k=i+1}^{n} (k-i)^{-1+\varepsilon} \geqslant C_\varepsilon n^\varepsilon.$$

The second integral is no larger than

$$\|b\|_1 \left\| \sum_{k=1}^{i} \left(\sin\left(2^k\pi\left(2^{-i}(-x-1)\right)\right)+1\right)' \right\|_\infty \leqslant C \sum_{k=1}^{i} 2^{k-i} \leqslant C$$

(since $\int b\, dx = 0$). Thus

$$\left| \int b(x)\mu_n\left(2^{-i}(-x-1)+1\right) dx \right| \geqslant C_\varepsilon n^\varepsilon - C \geqslant C'_\varepsilon n^\varepsilon,$$

if

(7) $$0 \leqslant i \leqslant n/2 \quad \text{and} \quad n > N_\varepsilon.$$

Therefore, if (7) holds,

$$b * \eta_{2^i\alpha_n}(2^i - 1) = \left(2^i\alpha_n\right)^{-1} n^{-2-\varepsilon_0} \int b(x)\mu_n\left(2^{-i}(2^i - 1 - x)\right) dx$$
$$\geqslant C'_\varepsilon \left(2^i\alpha_n\right)^{-1} n^{-2-\varepsilon_0+\varepsilon}.$$

Thus, $b^*_\eta(x) \geqslant C'_\varepsilon (2^i\alpha_n)^{-1} n^{-2-\varepsilon_0+\varepsilon}$ on $E_{n,i} = \{x\colon\ 2^{i-1}\alpha_n < |x| < 2^i\alpha_n - (2^i - 1)\}$. Thus,

$$\int_{E_{n,i}} b^*_\eta\, dx \geqslant C'_\varepsilon n^{-2-\varepsilon_0+\varepsilon},$$

which yields, upon summing for $0 < i \leqslant n/2$,

$$\int_{\alpha_n < |x| < 2^{n/2}\alpha_n} b^*_\eta\, dx \geqslant C'_\varepsilon n^{-1-\varepsilon_0+\varepsilon}.$$

Therefore

(8) $$\|b^*_\eta\|_{L^1} \geqslant C'_\varepsilon \sum_n n^{-1-\varepsilon_0+\varepsilon} = +\infty,$$

if $\varepsilon_0 < \varepsilon$.

Take $\nu(x) \in \mathcal{S}$ such that $\nu(x) + \eta(x) \geqslant 0$ for any $x \in R$. Then the kernel $\varphi = \nu + \eta$ is nonnegative and $a^*_\varphi \in L^1$ and $b^*_\varphi \notin L^1$, by (6) and (8). Thus

$$H^1_\varphi \neq \{0\} \quad \text{and} \quad H^1_\varphi \neq H^1.$$

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