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UNBOUNDED PERTURBATIONS OF FORCED HARMONIC OSCILLATIONS AT RESONANCE

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ABSTRACT. In 1969, A. C. Lazer and D. E. Leach proved an existence theorem for periodic solutions of Duffing's equations with bounded perturbations at resonance. In the present note, with the use of a topological technique, the author extended some results of Lazer and Leach to an n-dimensional Duffing system with unbounded perturbations at resonance.

1. In [**1**] A. C. Lazer proved an existence theorem for periodic solutions of a second-order equation of Duffing's type

$$d^2x/dt^2 + C(dx/dt) + g(x) = p(t), \tag{1.1}$$

where $g(x)$ is such that $xg(x) \geqslant 0$ for $|x|$ sufficiently large and $g(x)/x \to 0$ as $|x| \to \infty$. In 1972 J. Mawhin [**2**] extended Lazer's result to an n-dimensional Liénard system

$$\frac{d^2x}{dt^2} + \frac{d}{dt}[\text{grad } F(x)] + g(x) = p(t), \tag{1.2}$$

with conditions upon $g(x)$ analogous to those of Lazer (but in fact less severe).

In another paper [**3**], A. C. Lazer and D. E. Leach proved the existence of periodic solutions for Duffing's equation at resonance,

$$d^2x/dt^2 + m^2x + h(x) = p(t), \tag{1.3}$$

with bounded perturbations $h(x)$ and some other conditions.

In the present note, with the use of a topological technique used in [**5** and **6**], we propose to prove the existence of periodic solutions of an n-dimensional Duffing system at resonance,

$$d^2x_s/dt^2 + m_s^2x_s + f_s(t, x) = p_s(t) \tag{1.4}$$

$(s = 1, 2, \ldots, n)$ with unbounded perturbations $f_s(t, x)$ $(x = (x_1, x_2, \ldots, x_n))$ and some reasonable conditions, not much different from those in [**2, 3** and **4**]. It will always be assumed in the sequel that $m_s > 0$ are integers, $p_s(t) \in C(R, R)$ 2π-periodic in t, and $f_s(t, x)$ 2π-periodic in t and continuously differentiable with respect to x.

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2. Before proceeding to the proof of our main results we require some lemmas:

Let D_s be the closed unit disk centered at the origin O_s of the Euclidean plane R_s^2, for $s = 1, 2, \dots, n$. We then consider the topological product

$$G = D_1 \times D_2 \times \cdots \times D_n \subset R^{2n},$$

and its boundary $\partial G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n$, in which

$$\Gamma_1 = \partial D_1 \times D_2 \times \cdots \times D_n,$$
$$\Gamma_2 = D_1 \times \partial D_2 \times \cdots \times D_n, \dots, \Gamma_n = D_1 \times D_2 \times \cdots \times \partial D_n.$$

Denote by $\langle \cdot, \cdot \rangle$ the scalar product of vectors, and $|\cdot|$ the norm of a vector; let $z_s = (x_s, y_s)$ be a point or a vector in R_s^2, and then let $z = (z_1, z_2, \dots, z_n)$ be a point or a vector in R^{2n}.

Consider now a continuous mapping T: $G \to R^{2n}$ defined by $(z_1, z_2, \dots, z_n) \mapsto (w_1, w_2, \dots, w_n)$. Then the following lemma is an immediate consequence of a more general fixed-point theorem proved in [**6**].

Lemma 1. *If the above-mentioned mapping T satisfies the boundary condition*

$$\langle w_s - z_s, z_s \rangle \neq |z_s| \cdot |w_s - z_s|, \quad \text{for } z \in \Gamma_s \text{ and } s = 1, 2, \dots, n,$$

then T has at least one fixed point in G.

Next, by a similar method used in [**5**], we can derive an asymptotic formula of a definite integral containing a positive small parameter ε.

Lemma 2. *Let ε be a positive small parameter, A and $\delta < \frac{1}{2}$ be two positive constants, and*

$$J(\varepsilon) = \int_{-\pi/2+\sigma}^{\pi/2-\sigma} \cos\theta \cdot \tan^{-1}\left(\frac{1-\delta}{\varepsilon}\cos\theta\right) d\theta$$

where $\sigma = A\pi\varepsilon < \pi/2$. Then the asymptotic formula

$$J(\varepsilon) = \pi - \pi\varepsilon/(1-\delta) + O(\varepsilon^2)$$

holds if ε is small enough.

Corollary. *The asymptotic formula*

$$\int_{\pi/2+\sigma}^{3\pi/2-\sigma} \cos\theta \cdot \tan^{-1}\left(\frac{1-\delta}{\varepsilon}\cos\theta\right) d\theta = \pi - \frac{\pi\varepsilon}{1-\delta} + O(\varepsilon^2)$$

holds if ε is small enough.

3. To establish the existence theorems for periodic solutions of (1.4), we shall give some reasonable assumptions about the perturbations. The first one is a release from the boundedness restrictions on perturbations (cf. [**3, 2**]).

(H_1) There exists a positive constant A and a positive function $\beta(x)$ tending to zero as $|x| \to \infty$ such that the inequalities

$$|f_s(t, x)| \leqslant |x| \cdot \beta(x) \qquad (s = 1, 2, \dots, n),$$

hold for $t \in [0, 2\pi]$ and $|x| \geqslant A$.

Let $\lambda > 1$ be a large parameter, and $u_s = x_s/\lambda$ for $s = 1, 2, \ldots, n$. Then the differential system (1.4) becomes

$$d^2u_s/dt^2 + m_s^2 u_s + \lambda^{-1} f_s(t, \lambda u_s) = \lambda^{-1} p_s(t) \tag{3.1}$$

($s = 1, 2, \ldots, n$). Let us now consider an auxiliary system

$$d^2u_s/dt^2 + m_s^2 u_s + F_s(t, u, \varepsilon) = \varepsilon p_s(t) \tag{3.2}$$

where

$$F_s(t, u, \varepsilon) = \begin{cases} \varepsilon f_s(t, u/\varepsilon), & 0 < \varepsilon \leqslant 1; \\ 0, & \varepsilon = 0, \end{cases}$$

for $s = 1, 2, \ldots, n$.

It follows from (H_1) that $F_s(t, u, \varepsilon)$ is continuous with respect to (t, u, ε) and $F_s(t, u, 0) = 0$. Note that $F_s(t, u, \varepsilon)$ is 2π-periodic in t and differentiable with respect to u for any fixed ε ($0 \leqslant \varepsilon \leqslant 1$). Hence, we can still prove the continuity of solutions of (3.2) with respect to the parameter ε and the initial values.

In particular, when $\varepsilon = 1/\lambda$, the differential system (3.2) coincides with (3.1). We can put (3.2) in its equivalent form

$$\frac{du_s}{dt} = m_s v_s, \qquad \frac{dv_s}{dt} = -m_s u_s - \frac{1}{m_s} F_s(t, u, \varepsilon) + \frac{\varepsilon}{m_s} p_s(t) \tag{3.3}$$

($s = 1, 2, \ldots, n$). We then consider the initial conditions

$$\text{when } t = 0, u = \xi \quad \text{and} \quad v = \eta, \tag{3.4}$$

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ are two parameter n-vectors. It can be seen that the initial-value problem (3.3) and (3.4) has a unique solution,

$$u = u(t, \xi, \eta, \varepsilon), \quad v = v(t, \xi, \eta, \varepsilon) \qquad (-\infty < t < \infty), \tag{3.5}$$

which is continuous in $(t, \xi, \eta, \varepsilon)$ for all $t \in R$, $\xi \in R^n$, $\eta \in R^n$ and ε ($0 \leqslant \varepsilon \leqslant 1$). Therefore, for any fixed ε ($0 \leqslant \varepsilon \leqslant 1$), by setting

$$\bar{u} = u(2\pi, \xi, \eta, \varepsilon), \qquad \bar{v} = v(2\pi, \xi, \eta, \varepsilon), \tag{3.6}$$

we obtain a mapping $T_\varepsilon\colon G \to R^{2n}$, that is, $(\xi, \eta) \mapsto (\bar{u}, \bar{v})$ defined by (3.6), in which $(\xi, \eta) = (\xi_1, \eta_1, \xi_2, \eta_2, \ldots, \xi_n, \eta_n)$ and $(\bar{u}, \bar{v}) = (\bar{u}_1, \bar{v}_1, \bar{u}_2, \bar{v}_2, \ldots, \bar{u}_n, \bar{v}_n)$.

For any integer k ($1 \leqslant k \leqslant n$), let $(\xi, \eta) \in \Gamma_k$, and then we can set $\xi_k = \cos\theta_k^\circ$, $\eta_k = \sin\theta_k^\circ$ ($0 \leqslant \theta_k^\circ < 2\pi$) and

$$u_k(t, \xi, \eta, \varepsilon) = \gamma_k \cos\theta_k, \qquad v_k(t, \xi, \eta, \varepsilon) = \gamma_k \sin\theta_k,$$

where $\gamma_k = \gamma_k(t, \xi, \eta, \varepsilon) \geqslant 0$ and $\theta_k = \theta_k(t, \xi, \eta, \varepsilon)$ are continuous in $(t, \xi, \eta, \varepsilon)$. Obviously, we have

$$\gamma_k(t, \xi, \eta, 0) = 1, \qquad \theta_k(t, \xi, \eta, 0) = -m_k t + \theta_k^\circ. \tag{3.7}$$

From the continuity of γ_k and θ_k with respect to $(t, \xi, \eta, \varepsilon)$, together with (3.7), we conclude that, for any constant δ ($0 < \delta < \frac{1}{2}$), a positive number ε_1 ($\varepsilon_1 \leqslant 1$) can be found such that when $(\xi, \eta) \in \Gamma_k$, the inequality

$$1 - \delta \leqslant \gamma_k(t, \xi, \eta, \varepsilon) \leqslant 1 + \delta \tag{3.8}$$

holds for $0 \leqslant t \leqslant 4\pi$ and $0 \leqslant \varepsilon \leqslant \varepsilon_1$; and furthermore, we have

$$-2(m_k + 1)\pi < \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^\circ < -2(m_k - 1)\pi \tag{3.9}$$

for $0 \leqslant \varepsilon \leqslant \varepsilon_1$, and it follows from (3.3) that

$$\frac{d\theta_k}{dt} = -m_k - \frac{1}{m_k\gamma_k}F_k(t, u, \varepsilon)\cos\theta_k + \frac{\varepsilon}{m_k\gamma_k}p_k(t)\cos\theta_k. \tag{3.10}$$

LEMMA 3. *If* (H_1) *holds, then there is a positive constant* ε_2 *such that for any* ε $(0 \leqslant \varepsilon \leqslant \varepsilon_2)$ *and* $(\xi, \eta) \in \Gamma_k$,

$$d\theta_k/dt < 0 \quad \text{for } t \in [0, 4\pi] \qquad (k = 1, 2, \ldots, n).$$

(This lemma is an immediate consequence of (3.10) and (3.8) together with the continuity of $F(t, u, \varepsilon)$, $F(t, u, 0) = 0$.)

Geometrically, Lemma 3 means that for any ε $(0 \leqslant \varepsilon \leqslant \varepsilon_2)$ and $(\xi, \eta) \in \Gamma_k$, the projection $(u_k, v_k) \in R_k^2$ of the motion (3.5) moves clockwise around the origin O_k when $0 \leqslant t \leqslant 4\pi$.

Finally, we stress the inequality (3.9) by putting it in the following form:

LEMMA 4. *If* (H_1) *holds, then there is a positive constant* ε_1 *such that for any* ε $(0 \leqslant \varepsilon \leqslant \varepsilon_1)$ *and* $(\xi, \eta) \in \Gamma_k$, *the angular increment of the projection* $(u_k, v_k) \in R_k^2$ *of* (3.5),

$$\Delta\theta_k = \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^\circ = -2m_k\pi,$$

whenever $\Delta\theta_k$ *is an integral multiple of* 2π.

4. Now let us give the other basic assumptions.

(H_2) There exist positive constants A and B_j $(j = 1, 2, \ldots, n)$ such that when $t \in [0, 2\pi]$ and $|x_j| \geqslant A$, the inequalities

$$x_j f_j(t, x) \leqslant -B_j x_j \tan^{-1} x_j \qquad (j = 1, 2, \ldots, n)$$

hold.

(H_3) The constants B_j in (H_2) satisfy

$$B_j > E_j^\circ + F_j^\circ \qquad (j = 1, 2, \ldots, n),$$

where

$$E_j^\circ = \frac{1}{2\pi}\left|\int_0^{2\pi} p_j(t)\cos m_j t\, dt\right|, \qquad F_j^\circ = \frac{1}{2\pi}\left|\int_0^{2\pi} p_j(t)\sin m_j t\, dt\right|.$$

It is obvious that (H_3) implies the existence of a constant δ $(0 < \delta < \frac{1}{2})$ such that

$$\frac{1 - \delta}{(1 + \delta)^2}B_j > E_j^\circ + F_j^\circ \qquad (j = 1, 2, \ldots, n). \tag{4.1}$$

In what follows we shall take δ such that (4.1) holds.

LEMMA 5. *If* (H_1), (H_2) *and* (H_3) *hold, then there exists a positive constant* ε_3 *such that for any* ε $(0 < \varepsilon \leqslant \varepsilon_3)$ *and any* $(\xi, \eta) \in \Gamma_k$, *the increment*

$$\Delta\theta_k > -2m_k\pi \quad \text{for } k = 1, 2, \ldots, n.$$

PROOF. By Lemma 3, for any ε $(0 \leqslant \varepsilon \leqslant \varepsilon_2)$ and $(\xi, \eta) \in \Gamma_k$,

$$d\theta_k/dt = -\Phi_k < 0 \quad \text{for } 0 \leqslant t \leqslant 4\pi,$$

where

$$\Phi_k \equiv m_k + \frac{1}{m_k\gamma_k} F(t, u(t), \varepsilon)\cos\theta_k - \frac{\varepsilon}{m_k\gamma_k} p_k(t)\cos\theta_k.$$

It follows that from $\theta_k = \theta_k(t, \varepsilon)$ we can solve $t = t(\theta_k, \varepsilon)$ which is continuous in (θ_k, ε) and differentiable with respect to θ_k for any fixed ε $(0 \leqslant \varepsilon \leqslant \varepsilon_2)$. Then we can think of Φ_k as a continuous function of (θ_k, ε).

Given ε $(0 \leqslant \varepsilon \leqslant \varepsilon_2)$ and $(\xi, \eta) \in \Gamma_k$, let τ_k be the time such that the projection $(u_k, v_k) \in R_k^2$ of the motion (3.5) will complete m_k rotations around the origin O_k during the time interval $[0, \tau_k]$. Hence, by (3.10), we get

$$\tau_k = \int_{\theta_k^\circ - 2m_k\pi}^{\theta_k^\circ} \frac{d\theta_k}{\Phi_k},$$

which implies that $\tau_k = \tau_k(\varepsilon)$ is continuous in ε. We note that $\tau_k(0) = 2\pi$. Set

$$\tau_k = \sum_{j=0}^{m_k-1} \left(I_{1j} + I_{2j} + I_{3j} + I_{4j} + I_{5j}\right) \quad \text{for } 0 < \varepsilon \leqslant \varepsilon_2,$$

in which the integrals $I_{1j}, I_{2j}, I_{3j}, I_{4j}$ and I_{5j} of $1/\Phi_k$ are taken on the intervals

$$[-2j\pi - \pi/2 + \sigma, -2j\pi + \theta_k^\circ], \qquad [-2j\pi - \pi/2 - \sigma, -2j\pi - \pi/2 + \sigma],$$
$$[-2j\pi - 3\pi/2 + \sigma, -2j\pi - \pi/2 - \sigma], \qquad [-2j\pi - 3\pi/2 - \sigma, -2j\pi - 3\pi/2 + \sigma]$$

and

$$[-2(j+1)\pi + \theta_k^\circ, -2j\pi - 3\pi/2 - \sigma],$$

respectively, and $\sigma = A\pi\varepsilon$ and $j = 0, 1, \ldots, m_k - 1$. Here we assume $\theta_k^\circ \in [-\pi/2 + \sigma, \pi/2 - \sigma]$ for the definiteness, and the proof is also applicable to the other cases.

For simplicity, we set

$$\omega_k = \int_{\theta_k^\circ - 2m_k\pi}^{\theta_k^\circ} \frac{p_k(t)}{\gamma_k} \cos\theta_k\, d\theta_k = \sum_{j=0}^{m_k-1} \left(H_{1j} + H_{2j} + H_{3j} + H_{4j} + H_{5j}\right)$$

in which H_{ij} has the same interval of integration as that of I_{ij} $(i = 1, 2, 3, 4, 5;\ j = 0, 1, \ldots, m_k - 1)$, respectively.

For $1 - \delta \leqslant \gamma_k \leqslant 1 + \delta$ and

$$-2j\pi - \pi/2 + \sigma \leqslant \theta_k \leqslant -2j\pi + \theta_k^\circ \leqslant -2j\pi + \pi/2 - \sigma,$$

we have

$$\frac{u_k}{\varepsilon} = \frac{\gamma_k \cos\theta_k}{\varepsilon} \geqslant \frac{1 - \delta}{\varepsilon} \sin\sigma \geqslant \frac{2(1 - \delta)\sigma}{\pi\varepsilon} > \frac{\sigma}{\pi\varepsilon} = A.$$

Hence, by (H_2), it is not hard to derive the following inequalities:

$$I_{1j} > \frac{1}{m_k} \int_{-2j\pi-\pi/2+\sigma}^{-2j\pi+\theta_k^\circ} \left[1 - B_k \left(\frac{\varepsilon}{m_k \gamma_k} \right)^2 \frac{\gamma_k \cos\theta_k}{\varepsilon} \tan^{-1}\left(\frac{\gamma_k \cos\theta_k}{\varepsilon} \right) - \frac{\varepsilon}{m_k^2 \gamma_k} p_k(t) \cos\theta_k \right]^{-1} d\theta_k$$

$$> \frac{1}{m_k} \int_{-\pi/2+\sigma}^{\theta_k^\circ} \left[1 + \frac{\varepsilon(1-\delta) B_k}{(1+\delta)^2 m_k^2} \cos\theta_k \cdot \tan^{-1}\left(\frac{1-\delta}{\varepsilon} \cos\theta_k \right) \right] d\theta_k + \frac{\varepsilon}{m_k^3} H_{1j} + O(\varepsilon^2),$$

and

$$I_{5j} > \frac{1}{m_k} \int_{\theta_k^\circ}^{\pi/2-\sigma} \left[1 + \frac{\varepsilon(1-\delta) B_k}{(1+\delta)^2 m_k^2} \cos\theta_k \cdot \tan^{-1}\left(\frac{1-\delta}{\varepsilon} \cos\theta_k \right) \right] d\theta_k + \frac{\varepsilon}{m_k^3} H_{5j} + O(\varepsilon^2).$$

Then we have

$$I_{1j} + I_{5j} > \frac{1}{m_k} \int_{-\pi/2+\sigma}^{\pi/2-\sigma} \left[1 + \frac{\varepsilon(1-\delta) B_k}{(1+\delta)^2 m_k^2} \cos\theta_k \cdot \tan^{-1}\left(\frac{1-\delta}{\varepsilon} \cos\theta_k \right) \right] d\theta_k + \frac{\varepsilon}{m_k^3} (H_{1j} + H_{5j}) + O(\varepsilon^2)$$

$$= \frac{1}{m_k} \left[(\pi - 2\sigma) + \frac{\varepsilon(1-\delta) B_k}{(1+\delta)^2 m_k^2} J(\varepsilon) \right] + \frac{\varepsilon}{m_k^3} (H_{1j} + H_{5j}) + O(\varepsilon^2).$$

Hence, it follows from Lemma 2 that

$$I_{1j} + I_{5j} > \frac{1}{m_k} \left[(\pi - 2\sigma) + \frac{\varepsilon(1-\delta) B_k \pi}{(1+\delta)^2 m_k^2} \right] + \frac{\varepsilon}{m_k^3} (H_{1j} + H_{5j}) + O(\varepsilon^2).$$

By using the same technique and the corollary of Lemma 2, we have

$$I_{3j} > \frac{1}{m_k} \left[(\pi - 2\sigma) + \frac{\varepsilon(1-\delta) B_k \pi}{(1+\delta)^2 m_k^2} \right] + \frac{\varepsilon}{m_k^3} H_{3j} + O(\varepsilon^2).$$

On the other hand, since $H_{2j} = O(\varepsilon)$ and $H_{4j} = O(\varepsilon)$, we can write

$$I_{2j} = 2\sigma/m_k + (\varepsilon/m_k^3) H_{2j} + O(\varepsilon^2),$$

and

$$I_{4j} = 2\sigma/m_k + (\varepsilon/m_k^3) H_{4j} + O(\varepsilon^2).$$

We thus obtain the inequality

$$\tau_k > \frac{1}{m_k}\sum_{j=0}^{m_k-1}\left[2(\pi - 2\sigma) + \frac{2\varepsilon(1-\delta)B_k\pi}{(1+\delta)^2 m_k^2} + 4\sigma\right] + \frac{\varepsilon}{m_k^3}\omega_k + O(\varepsilon^2)$$

$$= 2\pi + \frac{2\varepsilon(1-\delta)B_k\pi}{(1+\delta)^2 m_k^2} + \frac{\varepsilon}{m_k^3}\omega_k + O(\varepsilon^2).$$

Since

$$\omega_k = \int_{\theta_k^\circ - 2m_k\pi}^{\theta_k^\circ} \frac{p_k(t)}{\gamma_k}\cos\theta_k\, d\theta_k = \int_0^{\tau_k} \frac{p_k(t)\cos\theta_k}{\gamma_k}\Phi_k\, dt$$

and

$$\lim_{\varepsilon\to 0}(\tau_k, \theta_k, \gamma_k, \Phi_k) = (2\pi, -m_k t + \theta_k^\circ, 1, m_k),$$

we get

$$\omega_k = \int_0^{2\pi} m_k p_k(t)\cos(-m_k t + \theta_k^\circ)\, dt + \zeta(\varepsilon)$$

$$\geqslant -2m_k\pi(E_k^\circ + F_k^\circ) + \zeta(\varepsilon),$$

where $\zeta(\varepsilon)$ is continuous in ε $(0 \leqslant \varepsilon \ll 1)$ and $\zeta(0) = 0$. Hence, we arrive at the inequality

$$\text{(4.2)} \qquad \tau_k > 2\pi + \frac{2\pi\varepsilon}{m_k^2}\left[\frac{(1-\delta)B_k}{(1+\delta)^2} - (E_k^\circ + F_k^\circ)\right] + \frac{\varepsilon\zeta(\varepsilon)}{m_k^3} + O(\varepsilon^2).$$

The inequalities (4.1) and (4.2) guarantee the existence of a positive constant ε_3 such that for any ε $(0 < \varepsilon \leqslant \varepsilon_3)$ and $(\xi, \eta) \in \Gamma_k$, the inequality $\tau_k > 2\pi$ is valid. It means that the time interval $[0, \tau_k]$, in which the projection (u_k, v_k) of the motion (3.5) will complete m_k rotations around the origin O_k, is longer than 2π. Hence, in the shorter interval $[0, 2\pi]$, the projection (u_k, v_k) can merely finish some rotations less than m_k, i.e.,

$$\Delta\theta_k = \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^\circ > -2m_k\pi,$$

and the proof of Lemma 5 is thus completed.

Finally, let us consider a variant form of (H_2).

(H_2^*) There exist positive constants A and B_j $(j = 1, 2, \ldots, n)$ such that when $t \in [0, 2\pi]$ and $|x_j| \geqslant A$, the inequalities

$$x_j f_j(t, x) \geqslant B_j x_j \tan^{-1} x_j \qquad (j = 1, 2, \ldots, n)$$

hold.

Then, in a similar way, we can prove the following lemma:

LEMMA 5*. *If* (H_1), (H_2^*) *and* (H_3) *hold, then there is a positive constant* ε_3^* *such that for any* ε $(0 < \varepsilon \leqslant \varepsilon_3^*)$ *and* $(\xi, \eta) \in \Gamma_k$, *the increment*

$$\Delta\theta_k = \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^\circ < -2m_k\pi \qquad (k = 1, 2, \ldots, n).$$

5. We are now in a position to prove our main theorems.

THEOREM 1. *If* (H_1), (H_2) *and* (H_3) *hold, then the differential system* (1.4) *has at least one* 2π*-periodic solution.*

PROOF. Consider the mapping

$$T_\varepsilon : G \to R^{2n} \qquad (0 < \varepsilon \ll 1),$$

defined in §3.

It follows from Lemmas 4 and 5 that

$$(\bar{u}_k - \xi_k)\xi_k + (\bar{v}_k - \eta_k)\eta_k \neq \sqrt{\xi_k^2 + \eta_k^2} \cdot \sqrt{(\bar{u}_k - \xi_k)^2 + (\bar{v}_k - \eta_k)^2}, \qquad (\xi, \eta) \in \Gamma_k$$

$(k = 1, 2, \ldots, n)$. Hence, by Lemma 1, we can conclude that T_ε has at least one fixed point, say (ξ^*, η^*), in G. Hence,

$$u = u(t, \xi^*, \eta^*, \varepsilon), \qquad v = v(t, \xi^*, \eta^*, \varepsilon)$$

is a 2π-periodic solution of (3.3), and then

$$x = \varepsilon^{-1} u(t, \xi^*, \eta^*, \varepsilon)$$

is a 2π-periodic solution of (1.4). The proof of Theorem 1 is thus completed.

By using Lemmas 4 and 5*, we can prove the following theorem:

THEOREM 2. *If* (H_1), (H_2^*) *and* (H_3) *hold, then the differential system* (1.4) *has at least one* 2π*-periodic solution.*

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