UNBOUNDED PERTURBATIONS
OF FORCED HARMONIC OSCILLATIONS AT RESONANCE

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Abstract. In 1969, A. C. Lazer and D. E. Leach proved an existence theorem for periodic solutions of Duffing's equations with bounded perturbations at resonance. In the present note, with the use of a topological technique, the author extended some results of Lazer and Leach to an n-dimensional Duffing system with unbounded perturbations at resonance.

1. In [1] A. C. Lazer proved an existence theorem for periodic solutions of a second-order equation of Duffing's type

\[ \frac{d^2x}{dt^2} + C\frac{dx}{dt} + g(x) = p(t), \]

where \( g(x) \) is such that \( xg(x) \geq 0 \) for \( |x| \) sufficiently large and \( g(x)/x \to 0 \) as \( |x| \to \infty \). In 1972 J. Mawhin [2] extended Lazer's result to an n-dimensional Liénard system

\[ \frac{d^2x}{dt^2} + \frac{d}{dt} \left[ \text{grad} \ F(x) \right] + g(x) = p(t), \]

with conditions upon \( g(x) \) analogous to those of Lazer (but in fact less severe).

In another paper [3], A. C. Lazer and D. E. Leach proved the existence of periodic solutions for Duffing's equation at resonance,

\[ \frac{d^2x}{dt^2} + m^2x + h(x) = p(t), \]

with bounded perturbations \( h(x) \) and some other conditions.

In the present note, with the use of a topological technique used in [5 and 6], we propose to prove the existence of periodic solutions of an n-dimensional Duffing system at resonance,

\[ \frac{d^2x_s}{dt^2} + m^2x_s + f_s(t, x) = p_s(t) \]

(\( s = 1, 2, \ldots, n \)) with unbounded perturbations \( f_s(t, x) \) (\( x = (x_1, x_2, \ldots, x_n) \)) and some reasonable conditions, not much different from those in [2, 3 and 4]. It will always be assumed in the sequel that \( m_x > 0 \) are integers, \( p_s(t) \in C(R, R) \) \( 2\pi \)-periodic in \( t \), and \( f_s(t, x) \) \( 2\pi \)-periodic in \( t \) and continuously differentiable with respect to \( x \).
2. Before proceeding to the proof of our main results we require some lemmas:

Let \( D_s \) be the closed unit disk centered at the origin \( O_s \) of the Euclidean plane \( \mathbb{R}^2 \), for \( s = 1, 2, \ldots, n \). We then consider the topological product

\[
G = D_1 \times D_2 \times \cdots \times D_n \subseteq \mathbb{R}^{2n},
\]
and its boundary \( \partial G = \Gamma_1 \cup \Gamma_2 \cup \cdots \cup \Gamma_n \), in which

\[
\Gamma_1 = \partial D_1 \times D_2 \times \cdots \times D_n, \\
\Gamma_2 = D_1 \times \partial D_2 \times \cdots \times D_n, \\
\vdots \\
\Gamma_n = D_1 \times D_2 \times \cdots \times \partial D_n.
\]

Denote by \( \langle \cdot, \cdot \rangle \) the scalar product of vectors, and \( |\cdot| \) the norm of a vector; let \( z_s = (x_s, y_s) \) be a point or a vector in \( \mathbb{R}^2 \), and then let \( z = (z_1, z_2, \ldots, z_n) \) be a point or a vector in \( \mathbb{R}^{2n} \).

Consider now a continuous mapping \( T: G \rightarrow \mathbb{R}^{2n} \) defined by \( (z_1, z_2, \ldots, z_n) \mapsto (w_1, w_2, \ldots, w_n) \). Then the following lemma is an immediate consequence of a more general fixed-point theorem proved in [6].

**Lemma 1.** If the above-mentioned mapping \( T \) satisfies the boundary condition

\[
\langle w_s - z_s, z_s \rangle \neq |z_s| \cdot |w_s - z_s|, \quad \text{for } z \in \Gamma_s \text{ and } s = 1, 2, \ldots, n,
\]

then \( T \) has at least one fixed point in \( G \).

Next, by a similar method used in [5], we can derive an asymptotic formula of a definite integral containing a positive small parameter \( \varepsilon \).

**Lemma 2.** Let \( \varepsilon \) be a positive small parameter, \( A \) and \( \delta < \frac{1}{2} \) be two positive constants, and

\[
J(\varepsilon) = \int_{-\pi/2}^{\pi/2 - \sigma} \cos \theta \cdot \tan^{-1}\left( \frac{1 - \delta}{\varepsilon} \cos \theta \right) d\theta
\]

where \( \sigma = A \pi \varepsilon < \pi/2 \). Then the asymptotic formula

\[
J(\varepsilon) = \pi - \pi \varepsilon / (1 - \delta) + O(\varepsilon^2)
\]

holds if \( \varepsilon \) is small enough.

**Corollary.** The asymptotic formula

\[
\int_{-\pi/2}^{\pi/2 - \sigma} \cos \theta \cdot \tan^{-1}\left( \frac{1 - \delta}{\varepsilon} \cos \theta \right) d\theta = \pi - \frac{\pi \varepsilon}{1 - \delta} + O(\varepsilon^2)
\]

holds if \( \varepsilon \) is small enough.

3. To establish the existence theorems for periodic solutions of (1.4), we shall give some reasonable assumptions about the perturbations. The first one is a release from the boundedness restrictions on perturbations (cf. [3, 2]).

\( (H_1) \) There exists a positive constant \( A \) and a positive function \( \beta(x) \) tending to zero as \( |x| \rightarrow \infty \) such that the inequalities

\[
|f_s(t, x)| \leq |x| \cdot \beta(x) \quad (s = 1, 2, \ldots, n),
\]

hold for \( t \in [0, 2\pi] \) and \( |x| \geq A \).
Let $\lambda > 1$ be a large parameter, and $u_s = x_s/\lambda$ for $s = 1, 2, \ldots, n$. Then the differential system (1.4) becomes

\begin{equation}
\frac{d^2 u_s}{dt^2} + m_s^2 u_s + \lambda^{-1} f_s(t, \lambda u_s) = \lambda^{-1} p_s(t)
\end{equation}

($s = 1, 2, \ldots, n$). Let us now consider an auxiliary system

\begin{equation}
\frac{d^2 u}{dt^2} + m_s^2 u_s + F_s(t, u, \epsilon) = \epsilon p_s(t)
\end{equation}

where

\[ F_s(t, u, \epsilon) = \begin{cases} \epsilon f_s(t, u/\epsilon), & 0 < \epsilon \leq 1; \\ 0, & \epsilon = 0, \end{cases} \]

for $s = 1, 2, \ldots, n$.

It follows from (H1) that $F_s(t, u, \epsilon)$ is continuous with respect to $(t, u, \epsilon)$ and $F_s(t, u, 0) = 0$. Note that $F_s(t, u, \epsilon)$ is $2\pi$-periodic in $t$ and differentiable with respect to $u$ for any fixed $\epsilon$ ($0 \leq \epsilon \leq 1$). Hence, we can still prove the continuity of solutions of (3.2) with respect to the parameter $\epsilon$ and the initial values.

In particular, when $\epsilon = 1/\lambda$, the differential system (3.2) coincides with (3.1). We can put (3.2) in its equivalent form

\begin{equation}
\begin{align*}
\frac{du_s}{dt} &= m_s v_s, \\
\frac{dv_s}{dt} &= -m_s u_s - \frac{1}{m_s} F_s(t, u, \epsilon) + \frac{\epsilon}{m_s} p_s(t)
\end{align*}
\end{equation}

($s = 1, 2, \ldots, n$). We then consider the initial conditions

\begin{equation}
\text{when } t = 0, u = \xi \quad \text{and} \quad v = \eta,
\end{equation}

where $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$ and $\eta = (\eta_1, \eta_2, \ldots, \eta_n)$ are two parameter $n$-vectors. It can be seen that the initial-value problem (3.3) and (3.4) has a unique solution,

\begin{equation}
u = u(t, \xi, \eta, \epsilon), \quad v = v(t, \xi, \eta, \epsilon)
\end{equation}

($-\infty < t < \infty$),

which is continuous in $(t, \xi, \eta, \epsilon)$ for all $t \in R, \xi \in R^n, \eta \in R^n$ and $\epsilon$ ($0 \leq \epsilon \leq 1$). Therefore, for any fixed $\epsilon$ ($0 \leq \epsilon \leq 1$), by setting

\begin{equation}
\begin{align*}
\tilde{u} &= u(2\pi, \xi, \eta, \epsilon), \\
\tilde{v} &= v(2\pi, \xi, \eta, \epsilon),
\end{align*}
\end{equation}

we obtain a mapping $T_\epsilon: G \to R'^n$, that is, $(\xi, \eta) \mapsto (\tilde{u}, \tilde{v})$ defined by (3.6), in which $(\xi, \eta) = (\xi_1, \xi_2, \eta_1, \eta_2, \ldots, \xi_n, \eta_n)$ and $(\tilde{u}, \tilde{v}) = (\tilde{u}_1, \tilde{v}_1, \tilde{u}_2, \tilde{v}_2, \ldots, \tilde{u}_n, \tilde{v}_n)$.

For any integer $k$ ($1 \leq k \leq n$), let $(\xi, \eta) \in \Gamma_k$, and then we can set $\xi_k = \cos \theta_k^0, \eta_k = \sin \theta_k^0$ ($0 \leq \theta_k^0 < 2\pi$) and

\begin{equation}
\begin{align*}
u_k(t, \xi, \eta, \epsilon) &= \gamma_k \cos \theta_k^0, \\
v_k(t, \xi, \eta, \epsilon) &= \gamma_k \sin \theta_k^0,
\end{align*}
\end{equation}

where $\gamma_k = \gamma_k(t, \xi, \eta, \epsilon) \geq 0$ and $\theta_k = \theta_k(t, \xi, \eta, \epsilon)$ are continuous in $(t, \xi, \eta, \epsilon)$. Obviously, we have

\begin{equation}
\gamma_k(t, \xi, \eta, 0) = 1, \quad \theta_k(t, \xi, \eta, 0) = -m_k t + \theta_k^0.
\end{equation}

From the continuity of $\gamma_k$ and $\theta_k$ with respect to $(t, \xi, \eta, \epsilon)$, together with (3.7), we conclude that, for any constant $\delta$ ($0 < \delta < 1/2$), a positive number $\epsilon_1$ ($\epsilon_1 \leq 1$) can be found such that when $(\xi, \eta) \in \Gamma_k$, the inequality

\begin{equation}
1 - \delta \leq \gamma_k(t, \xi, \eta, \epsilon) \leq 1 + \delta
\end{equation}
holds for \(0 \leq t \leq 4\pi\) and \(0 \leq \varepsilon \leq \varepsilon_1\); and furthermore, we have
\[
-2(m_k + 1)\pi < \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^0 < -2(m_k - 1)\pi
\]
for \(0 \leq \varepsilon \leq \varepsilon_1\), and it follows from (3.3) that
\[
\frac{d\theta_k}{dt} = -m_k - \frac{1}{m_k\gamma_k} F_k(t, u, \varepsilon)\cos\theta_k + \frac{\varepsilon}{m_k\gamma_k} p_k(t)\cos\theta_k.
\]

**Lemma 3.** If \((H_1)\) holds, then there is a positive constant \(\varepsilon_2\) such that for any \(\varepsilon\) \((0 \leq \varepsilon \leq \varepsilon_2)\) and \((\xi, \eta) \in \Gamma_k\),
\[
\frac{d\theta_k}{dt} < 0 \quad \text{for} \quad t \in [0, 4\pi] \quad (k = 1, 2, \ldots, n).
\]

(This lemma is an immediate consequence of (3.10) and (3.8) together with the continuity of \(F(t, u, \varepsilon), F(t, u, 0) = 0\).)

Geometrically, Lemma 3 means that for any \(\varepsilon\) \((0 \leq \varepsilon \leq \varepsilon_2)\) and \((\xi, \eta) \in \Gamma_k\), the projection \((u_k, v_k) \in R^2_k\) of the motion (3.5) moves clockwise around the origin \(O_k\) when \(0 \leq t \leq 4\pi\).

Finally, we stress the inequality (3.9) by putting it in the following form:

**Lemma 4.** If \((H_1)\) holds, then there is a positive constant \(\varepsilon_1\) such that for any \(\varepsilon\) \((0 \leq \varepsilon \leq \varepsilon_1)\) and \((\xi, \eta) \in \Gamma_k\), the angular increment of the projection \((u_k, v_k) \in R^2_k\) of (3.5),
\[
\Delta\theta_k = \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^0 = -2m_k\pi,
\]
whenever \(\Delta\theta_k\) is an integral multiple of \(2\pi\).

4. Now let us give the other basic assumptions.

\((H_2)\) There exist positive constants \(A\) and \(B_j\) \((j = 1, 2, \ldots, n)\) such that when \(t \in [0, 2\pi]\) and \(|x_j| \geq A\), the inequalities
\[
x_j f_j(t, x) \leq -B_j x_j \tan^{-1} x_j \quad (j = 1, 2, \ldots, n)
\]
hold.

\((H_3)\) The constants \(B_j\) in \((H_2)\) satisfy
\[
B_j > E_j^\circ + F_j^\circ \quad (j = 1, 2, \ldots, n),
\]
where
\[
E_j^\circ = \frac{1}{2\pi} \left|\int_0^{2\pi} p_j(t)\cos m_j t \, dt\right|, \quad F_j^\circ = \frac{1}{2\pi} \left|\int_0^{2\pi} p_j(t)\sin m_j t \, dt\right|.
\]

It is obvious that \((H_3)\) implies the existence of a constant \(\delta\) \((0 < \delta < \frac{1}{2})\) such that
\[
\frac{1}{(1 + \delta)^2} B_j > E_j^\circ + F_j^\circ \quad (j = 1, 2, \ldots, n).
\]

In what follows we shall take \(\delta\) such that (4.1) holds.

**Lemma 5.** If \((H_1), (H_2)\) and \((H_3)\) hold, then there exists a positive constant \(\varepsilon_3\) such that for any \(\varepsilon\) \((0 \leq \varepsilon \leq \varepsilon_3)\) and any \((\xi, \eta) \in \Gamma_k\), the increment
\[
\Delta\theta_k > -2m_k\pi \quad \text{for} \quad k = 1, 2, \ldots, n.
\]
PROOF. By Lemma 3, for any \( \varepsilon \) (0 \( \leq \) \( \varepsilon \) \( \leq \) \( \varepsilon _2 \)) and \((\xi, \eta) \in \Gamma_k\),
\[
d\theta_k/\!d\!t = -\Phi_k < 0 \quad \text{for} \quad 0 \leq t \leq 4\pi,
\]
where
\[
\Phi_k \equiv m_k + \frac{1}{m_k\gamma_k} F(t, u(t), \varepsilon)\cos \theta_k - \frac{\varepsilon}{m_k\gamma_k} p_k(t) \cos \theta_k.
\]
It follows that from \( \theta_k = \theta_k(t, \varepsilon) \) we can solve \( t = t(\theta_k, \varepsilon) \) which is continuous in \((\theta_k, \varepsilon)\) and differentiable with respect to \( \theta_k \) for any fixed \( \varepsilon \) (0 \( \leq \varepsilon \) \( \leq \varepsilon _2 \)). Then we can think of \( \Phi_k \) as a continuous function of \((\theta_k, \varepsilon)\).

Given \( \varepsilon \) (0 \( \leq \varepsilon \) \( \leq \varepsilon _2 \)) and \((\xi, \eta) \in \Gamma_k\), let \( \tau_k \) be the time such that the projection \((u_k, v_k) \in R_k^2\) of the motion (3.5) will complete \( m_k \) rotations around the origin \( O_k \) during the time interval \([0, \tau_k]\). Hence, by (3.10), we get
\[
\tau_k = \int_{\theta_k - 2m_k\pi}^{\theta_k} \frac{d\theta_k}{\Phi_k},
\]
which implies that \( \tau_k = \tau_k(\varepsilon) \) is continuous in \( \varepsilon \). We note that \( \tau_k(0) = 2\pi \). Set
\[
\tau_k = \sum_{j=0}^{m_k-1} (I_{1j} + I_{2j} + I_{3j} + I_{4j} + I_{5j}) \quad \text{for} \quad 0 < \varepsilon \leq \varepsilon _2,
\]
in which the integrals \( I_{1j}, I_{2j}, I_{3j}, I_{4j} \) and \( I_{5j} \) of \( 1/\Phi_k \) are taken on the intervals
\[
[-2j\pi - \pi/2 + \alpha, -2j\pi + \theta_k^\circ], \quad [-2j\pi - \pi/2 - \sigma, -2j\pi - \pi/2 + \sigma],
\]
\[
[-2j\pi - 3\pi/2 + \alpha, -2j\pi - \pi/2 - \sigma], \quad [-2j\pi - 3\pi/2 - \sigma, -2j\pi - 3\pi/2 + \sigma]
\]
and
\[
[-2(j + 1)\pi + \theta_k^\circ, -2j\pi - 3\pi/2 - \sigma],
\]
respectively, and \( \sigma = A\pi\varepsilon \) and \( j = 0, 1, \ldots, m_k - 1 \). Here we assume \( \theta_k^\circ \in [-\pi/2 + \alpha, \pi/2 - \sigma] \) for the definiteness, and the proof is also applicable to the other cases.

For simplicity, we set
\[
\omega_k = \int_{\theta_k^\circ - 2m_k\pi}^{\theta_k^\circ} \frac{p_k(t)}{\gamma_k} \cos \theta_k \, d\theta_k = \sum_{j=0}^{m_k-1} (H_{1j} + H_{2j} + H_{3j} + H_{4j} + H_{5j}),
\]
in which \( H_{ij} \) has the same interval of integration as that of \( I_{ij} (i = 1, 2, 3, 4, 5; j = 0, 1, \ldots, m_k - 1) \), respectively.

For \( 1 - \delta \leq \gamma_k \leq 1 + \delta \) and
\[
-2j\pi - \pi/2 + \sigma \leq \theta_k \leq -2j\pi + \theta_k^\circ \leq -2j\pi + \pi/2 - \sigma,
\]
we have
\[
\frac{u_k}{\varepsilon} = \frac{\gamma_k}{\varepsilon} \cos \frac{\theta_k}{\varepsilon} \geq \frac{1 - \delta}{\varepsilon} \sin \sigma \geq \frac{2(1 - \delta)\sigma}{\pi\varepsilon} > \frac{\sigma}{\pi\varepsilon} = A.
\]
Hence, by \((H_2)\), it is not hard to derive the following inequalities:

\[
I_{1j} > \frac{1}{m_k} \int_{-\pi/2+\sigma}^{\pi/2-\sigma} \left[ 1 - B_k \left( \frac{\epsilon}{m_k \gamma_k} \right)^2 \gamma_k \cos \theta_k \frac{\tan^{-1} \left( \frac{\gamma_k \cos \theta_k}{\epsilon} \right)}{1 - B_k} \right] d\theta_k \\
> \frac{1}{m_k^3} \int_{-\pi/2+\sigma}^{\pi/2-\sigma} \left[ 1 + \frac{\epsilon (1 - \delta) B_k}{(1 + \delta)^2 m_k^2} \cos \theta_k \cdot \tan^{-1} \left( \frac{1 - \delta}{\epsilon} \cos \theta_k \right) \right] \cos \theta_k \cdot \tan^{-1} \left( \frac{1 - \delta}{\epsilon} \cos \theta_k \right) d\theta_k \\
+ \frac{\epsilon}{m_k^3} H_{1j} + O(\epsilon^2),
\]

and

\[
I_{5j} > \frac{1}{m_k} \int_{-\pi/2+\sigma}^{\pi/2-\sigma} \left[ 1 + \frac{\epsilon (1 - \delta) B_k}{(1 + \delta)^2 m_k^2} \cos \theta_k \cdot \tan^{-1} \left( \frac{1 - \delta}{\epsilon} \cos \theta_k \right) \right] d\theta_k \\
+ \frac{\epsilon}{m_k^3} H_{5j} + O(\epsilon^2).
\]

Then we have

\[
I_{1j} + I_{5j} > \frac{1}{m_k} \int_{-\pi/2+\sigma}^{\pi/2-\sigma} \left[ 1 + \frac{\epsilon (1 - \delta) B_k}{(1 + \delta)^2 m_k^2} \cos \theta_k \cdot \tan^{-1} \left( \frac{1 - \delta}{\epsilon} \cos \theta_k \right) \right] d\theta_k \\
+ \frac{\epsilon}{m_k^3} \left( H_{1j} + H_{5j} \right) + O(\epsilon^2) \\
= \frac{1}{m_k} \left[ (\pi - 2\sigma) + \frac{\epsilon (1 - \delta) B_k \pi}{(1 + \delta)^2 m_k^2} J(\epsilon) \right] + \frac{\epsilon}{m_k^3} \left( H_{1j} + H_{5j} \right) + O(\epsilon^2).
\]

Hence, it follows from Lemma 2 that

\[
I_{1j} + I_{5j} > \frac{1}{m_k} \left[ (\pi - 2\sigma) + \frac{\epsilon (1 - \delta) B_k \pi}{(1 + \delta)^2 m_k^2} \right] + \frac{\epsilon}{m_k^3} \left( H_{1j} + H_{5j} \right) + O(\epsilon^2).
\]

By using the same technique and the corollary of Lemma 2, we have

\[
I_{3j} > \frac{1}{m_k} \left[ (\pi - 2\sigma) + \frac{\epsilon (1 - \delta) B_k \pi}{(1 + \delta)^2 m_k^2} \right] + \frac{\epsilon}{m_k^3} H_{3j} + O(\epsilon^2).
\]

On the other hand, since \(H_{2j} = O(\epsilon)\) and \(H_{4j} = O(\epsilon)\), we can write

\[
I_{2j} = 2\sigma/m_k + \left( \epsilon/m_k^3 \right) H_{2j} + O(\epsilon^2),
\]

and

\[
I_{4j} = 2\sigma/m_k + \left( \epsilon/m_k^3 \right) H_{4j} + O(\epsilon^2).
\]
We thus obtain the inequality
\[
\tau_k > \frac{1}{m_k} \sum_{j=0}^{m_k-1} \left[ 2(\pi - 2\sigma) + \frac{2\varepsilon(1 - \delta)B_k\pi}{(1 + \delta)^2m_k^2} + 4\sigma \right] + \frac{\varepsilon}{m_k^3} \omega_k + O(\varepsilon^2)
\]
\[
= 2\pi + \frac{2\varepsilon(1 - \delta)B_k\pi}{(1 + \delta)^2m_k^2} + \frac{\varepsilon}{m_k^3} \omega_k + O(\varepsilon^2).
\]

Since
\[
\omega_k = \int_{\theta_k^* - 2m_k\pi}^{\theta_k^*} \frac{p_k(t)}{\gamma_k} \cos \theta_k \, d\theta_k = \int_0^{\tau_k} \frac{p_k(t)}{\gamma_k} \cos \theta_k \, d\theta_k dt
\]
and
\[
\lim_{\varepsilon \to 0} (\tau_k, \theta_k, \gamma_k, \Phi_k) = (2\pi, -m_k t + \theta_k^o, 1, m_k),
\]
we get
\[
\omega_k = \int_0^{2\pi} m_k p_k(t) \cos(-m_k t + \theta_k^o) \, dt + \xi(\varepsilon)
\]
\[
\geq -2m_k \pi\left( E_k^o + F_k^o \right) + \xi(\varepsilon),
\]
where \(\xi(\varepsilon)\) is continuous in \(\varepsilon\) (0 < \(\varepsilon\) < 1) and \(\xi(0) = 0\). Hence, we arrive at the inequality
\[
(4.2) \quad \tau_k > 2\pi + \frac{2\pi\varepsilon}{m_k^2} \left( \frac{(1 - \delta)B_k}{(1 + \delta)^2} - (E_k^o + F_k^o) \right) + \frac{\varepsilon\xi(\varepsilon)}{m_k^3} + O(\varepsilon^2).
\]

The inequalities (4.1) and (4.2) guarantee the existence of a positive constant \(\varepsilon_3\) such that for any \(\varepsilon\) (0 < \(\varepsilon\) < \(\varepsilon_3\)) and \((\xi, \eta) \in \Gamma_k\), the inequality \(\tau_k > 2\pi\) is valid. It means that the time interval \([0, \tau_k]\), in which the projection \((u_k, v_k)\) of the motion (3.5) will complete \(m_k\) rotations around the origin \(O_k\), is longer than \(2\pi\). Hence, in the shorter interval \([0, 2\pi]\), the projection \((u_k, v_k)\) can merely finish some rotations less than \(m_k\), i.e.,
\[
\Delta \theta_k = \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^o > -2m_k\pi,
\]
and the proof of Lemma 5 is thus completed.

Finally, let us consider a variant form of (H2).

(Hf) There exist positive constants \(A\) and \(B_j\) (\(j = 1, 2, \ldots, n\)) such that when \(t \in [0, 2\pi]\) and \(|x_j| \geq A\), the inequalities
\[
x_j f_j(t, x) \geq B_j x_j \tan^{-1} x_j \quad (j = 1, 2, \ldots, n)
\]
hold.

Then, in a similar way, we can prove the following lemma:

**Lemma 5*. If (H1), (Hf) and (H3) hold, then there is a positive constant \(\varepsilon_3^*\) such that for any \(\varepsilon\) (0 < \(\varepsilon\) < \(\varepsilon_3^*\)) and \((\xi, \eta) \in \Gamma_k\), the increment
\[
\Delta \theta_k = \theta_k(2\pi, \xi, \eta, \varepsilon) - \theta_k^o < -2m_k\pi \quad (k = 1, 2, \ldots, n).
\]

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5. We are now in a position to prove our main theorems.

**Theorem 1.** If \((H_1), (H_2)\) and \((H_3)\) hold, then the differential system \((1.4)\) has at least one \(2\pi\)-periodic solution.

**Proof.** Consider the mapping

\[ T_\varepsilon : G \to R^{2n} \quad (0 < \varepsilon \ll 1), \]

defined in §3.

It follows from Lemmas 4 and 5 that

\[ (\bar{u}_k - \xi_k)\xi_k + (\bar{v}_k - \eta_k)\eta_k \]

\[ \neq \sqrt{\xi_k^2 + \eta_k^2} \cdot \sqrt{(\bar{u}_k - \xi_k)^2 + (\bar{v}_k - \eta_k)^2}, \quad (\xi, \eta) \in \Gamma_k \]

\((k = 1, 2, \ldots, n)\). Hence, by Lemma 1, we can conclude that \(T_\varepsilon\) has at least one fixed point, say \((\xi^*, \eta^*)\), in \(G\). Hence,

\[ u = u(t, \xi^*, \eta^*, \varepsilon), \quad v = v(t, \xi^*, \eta^*, \varepsilon) \]

is a \(2\pi\)-periodic solution of \((3.3)\), and then

\[ x = \varepsilon^{-1}u(t, \xi^*, \eta^*, \varepsilon) \]

is a \(2\pi\)-periodic solution of \((1.4)\). The proof of Theorem 1 is thus completed.

By using Lemmas 4 and 5*, we can prove the following theorem:

**Theorem 2.** If \((H_1), (H_2^*)\) and \((H_3)\) hold, then the differential system \((1.4)\) has at least one \(2\pi\)-periodic solution.

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**References**


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