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ON HARDY'S INEQUALITY IN WEIGHTED REARRANGEMENT INVARIANT SPACES AND APPLICATIONS. I

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ABSTRACT. We give inequalities relating the norm of a function and the norm of its average operators P_ψ, Q_ψ and S_ψ, T_ψ in weighted rearrangement invariant spaces $E_{\kappa,\delta}$ and $E(\mu)$, $d\mu(t) = \tau'(t)\,dt$. These average operators include, for example, the integral mean, the P_p, Q_p operators of Boyd [4] and Butzer and Fehér [6], the average operators P_φ, Q_φ and S_E, T_E from [14, 15, 16]. In the particular case, for some ψ, κ, δ, τ and E these inequalities were obtained by many authors and applied to a study of interpolation operators and imbedding theorems for Sobolev weight spaces.

Let ψ be a positive nondecreasing continuous function on $I = (0, l)$, $0 < l \leqslant \infty$. We define the *average operators* P_ψ, Q_ψ and (if additionally, ψ is an increasing absolutely continuous function with $\psi(0^+) = \lim_{t\to 0^+} \psi(t) = 0$) S_ψ, T_ψ, namely

$$(1) \qquad (P_\psi x)(t) := \psi(t)^{-1}\int_0^t x(s)\psi(s)s^{-1}\,ds,$$

$$(2) \qquad (Q_\psi x)(t) := \psi(t)^{-1}\int_t^l x(s)\psi(s)s^{-1}\,ds,$$

$$(3) \qquad (S_\psi x)(t) := \psi(t)^{-1}\int_0^t x(s)\psi'(s)\,ds,$$

$$(4) \qquad (T_\psi x)(t) := \psi(t)^{-1}\int_t^l x(s)\psi'(s)\,ds$$

whenever the required integral exists for almost all $t \in I$. For $\psi(t) = t^a$, $0 < a < \infty$, we write P_a, Q_a and S_a, T_a, instead of P_ψ, Q_ψ and S_ψ, T_ψ, respectively.

We will investigate the boundedness of operators (1)–(4) in weighted rearrangement invariant spaces with the aid of indices.

The lower index $p^l(\psi)$ and the upper index $q^l(\psi)$ of a positive measurable function ψ on I are defined in terms of the submultiplicative function on $(0, \infty)$,

$$M^l(s,\psi) = \sup_{t\in I,\, ts\in I} \frac{\psi(ts)}{\psi(t)} = \sup_{t\in(0,\min(1,1/s)l)} \frac{\psi(ts)}{\psi(t)}:$$

$$(5) \qquad p^l(\psi) = \lim_{s\to 0^+} \frac{\ln M^l(s,\psi)}{\ln s},$$

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(6) $$q^l(\psi) = \lim_{s\to\infty} \frac{\ln M^l(s,\psi)}{\ln s}$$

(see [**16, 10**]).

REMARK 1. We shall consider only ψ such that limits (5), (6) exist. We see that this is meaningful if $M^l(s,\psi)$ is a finite measurable function on $(0,\infty)$ or $M^l(s,\psi)$ is a monotone function on $(0,\infty)$.

We assume that $E = E(m)$ is an ideal Banach function space of measurable (equivalence classes of Lebesgue measurable) functions on $I = (0, l)$, $0 < l \leqslant \infty$, which norm $\|\cdot\|_E$ is rearrangement invariant in the sense that two functions which are equimeasurable with respect to m have the same norm. We called this space a *rearrangement invariant space* (r.i. space) or *symmetric space* in the terminology of Semenov (for more details, see [**10**]).

Examples of r.i. spaces include the Lebesgue L^p-spaces, the Orlicz L^F-spaces, the Lorentz spaces Λ, M, $L(p,q)$ and the Lorentz-Orlicz spaces $L(\varphi, L^F)$, $\Lambda(\varphi, L^F)$. Also if E, F are r.i. spaces, so are $E \cap F$, $E + F$ and $\overline{A}^E$, where A is symmetric linear subset of E (A is *symmetric* if $x \in A$ and $y^*(t) \leqslant x^*(t)$ imply $y \in A$).

If X and Y are (quasi-)Banach spaces, $[X, Y]$ will denote the space of all bounded linear operators from X into Y. We abbreviate $[X, X] = [X]$.

The lower index $\alpha^l(E)$ and the upper index $\beta^l(E)$ corresponding to the r.i. space E on I were defined by Boyd [**4**] in terms of the submultiplicative function $h(s, E)$, where, for r.i. space E, $h(s,E)$ denotes the norm in $[E]$ of the dilation operator σ_s, $(\sigma_s x)(t) = x(t/s)$ if $t/s \leqslant l$, $(\sigma_s x)(t) = 0$ if $t/s > l$:

(7) $$\alpha^l(E) = \lim_{s\to 0^+} \frac{\ln h(s,E)}{\ln s},$$

(8) $$\beta^l(E) = \lim_{s\to\infty} \frac{\ln h(s,E)}{\ln s}.$$

For the various r.i. spaces E and functions ψ the indices were computed by Boyd [**5**] and the author [**15, 16**].

The weighted r.i. space $E_{\kappa,\delta}$, where κ is a positive measurable function on I and δ: $I \to I$ is a measurable function, consists of those measurable functions x on I for which $x \circ \delta$ is measurable and

(9) $$\|x\|_{E_{\kappa,\delta}} = \|\kappa(t)x(\delta(t))\|_E < \infty.$$

Let us denote by $E^{(*)}_{\kappa,\delta}$ the set of all measurable functions x on I for which $\kappa(t)x^*(\delta(t)) \in E$. The set $E^{(*)}_{\kappa,\delta}$ is not a Banach space in general (functional $\|\kappa(t)x^*(\delta(t))\|_E$ does not define a norm, since the triangle inequality fails); therefore we introduce the r.i. space $E^{(**)}_{\kappa,\delta}$ with the norm

(10) $$\|x\|_{E^{(**)}_{\kappa,\delta}} = \|\kappa(t)x^{**}(\delta(t))\|_E, \quad \text{where } x^{**}(t) = \frac{1}{t}\int_0^t x^*(s)\,ds.$$

REMARK 2. One has

(11) $$E^{(**)}_{\kappa,\delta} \subset E^{(*)}_{\kappa,\delta}$$

with a continuous inclusion map. If equality in (11) holds, then $E^{(*)}_{\kappa,\delta}$ can be renormed equivalently.

EXAMPLE 1. Let τ: $I \to I$ be an increasing, absolutely continuous function with $\tau(I) = I$. For any measurable set $A \subset I$ we define measure by

$$\mu(A) = \int_A \tau'(t)\,dt. \tag{12}$$

The weighted r.i. space $E(\mu)$ then consists of those μ-measurable functions on I for which $x^*_\mu \in E(m)$ and the norm in $E(\mu)$ is given by $\|x\|_{E(\mu)} = \|x^*_\mu\|_E$. Here, as usual, x^*_μ denotes the nonnegative, nonincreasing rearrangement of $|x|$ which is equimeasurable with $|x|$ in the sense that

$$\mu\{t \in I: |x(t)| > a\} = m\{t \in I: x^*_\mu > a\} \tag{13}$$

for all $a > 0$.

PROPOSITION 1. *$E(\mu) = E_{1,\tau^{-1}}$ with equal norms.*

PROOF. First, we prove that, for any measurable nonnegative function x on I,

$$\mu\{t \in I: x(t) > a\} = m\{t \in I: x(t^{-1}(t)) > a\} \tag{14}$$

for all $a > 0$.

For any measurable set $A \subset I$ we have $\mu(A) = m(\tau(A))$ (see [**13**, p. 174]). Applying this to

$$A = \{t \in I: x(t) > a\} = x^{-1}(a, \infty) = \tau^{-1}[\tau(x^{-1}(a,\infty))] = \tau^{-1}(B),$$
$$B = \tau(x^{-1}(a,\infty)) = \{t \in I: x(\tau^{-1}(t)) > a\}$$

we get

$$\mu(A) = m(\tau(\tau^{-1}(B))) = m(B).$$

From (14), we have, for all $a > 0$,

$$m\{t \in I: x^*_\mu > a\} \overset{(13)}{=} \mu\{t \in I: |x(t)| > a\} \overset{(14)}{=} m\{t \in I: |x(\tau^{-1}(t))| > a\}.$$

Hence $\|x^*_\mu\|_E = \|x \circ \tau^{-1}\|_E$, i.e.

$$\|x\|_{E(\mu)} = \|x^*_\mu\|_E = \|x \circ \tau_\varphi^{-1}\|_E = \|x\|_{E_{1,\tau^{-1}}}.$$

EXAMPLE 2. Let $1 \leqslant q < \infty$ and let φ be a positive, nondecreasing, concave function on I such that $\varphi(0^+) = 0$. Then $L^{q(*)}_{\varphi'(t)^{1/q},t} = \Lambda(\varphi, q)$ and $L^{q(*)}_{t^{-1/q}\varphi(t),t} = L(\varphi, q)$, where

$$\|x\|_{\Lambda(\varphi,q)} = \left(\int_0^l x^*(t)^q \varphi'(t)\,dt\right)^{1/q}, \tag{15}$$

$$\|x\|_{L(\varphi,q)} = \left(\int_0^l [\varphi(t)x^*(t)]^q\,dt/t\right)^{1/q}. \tag{16}$$

Note that $\|\cdot\|_{\Lambda(\varphi,q)}$ satisfies the triangle inequality and $\|\cdot\|_{L(\varphi,q)}$ satisfies the triangle inequality iff $t^{-1/q}\varphi(t)$ is equivalent to a nonincreasing function (Lorentz **[12]**). However, if $q^l(\varphi) < 1$ then in $L(\varphi, q)$ an equivalent norm can be introduced (see Proposition 2).

LEMMA 1 [**10**, p. 136]. *If E is a r.i. space on I, φ a nondecreasing continuous function, and u a nonnegative measurable function on (a, b), $0 \leqslant a < b \leqslant \infty$, then, for all $x \in E$,*

$$\left\| \int_a^b \sigma_{1/s} x^*(t) u(s)\, d\varphi(s) \right\|_E \leqslant \int_a^b \|\sigma_{1/s} x^*(t)\|_E u(s)\, d\varphi(s). \tag{17}$$

For a positive nondecreasing function f defined on I and for r.i. space E on I we denote

$$0 \leqslant a_f \leqslant tf'(t)/f(t) \leqslant b_f \leqslant \infty \quad \text{a.e. on } I, \tag{18}$$

$$C_0(E, f) = \int_0^1 h(s^{-1}, E) M^l(s, f)\, ds/s. \tag{19}$$

THEOREM 1. *Let E be a r.i. space on I and let $\psi\colon I \to \mathbf{R}_+$ be a nondecreasing continuous function, $\delta\colon I \to I$ a nondecreasing absolutely continuous function, and $\kappa\colon I \to \mathbf{R}_+$ a measurable function such that $\delta(0^+) = 0$ and $p^l(\psi \circ \delta/\kappa) > 0$.*

(a) *If $b_\delta < \infty$, E has Fatou norm and $\beta^l(E) < p^l(\psi \circ \delta/\kappa)$, then $P_\psi \in [E_{\kappa,\delta}]$ and $\|P_\psi\| \leqslant b_\delta C_0(E, \psi \circ \delta/\kappa)$.*

(b) *If $b_\delta < \infty$ and $\beta^l(E) < \min\{1, p^l(\psi \circ \delta/\kappa)\}$, then $P_\psi \in [E_{\kappa,\delta}]$ and $\|P_\psi\| \leqslant b_\delta C_0(E, \psi \circ \delta/\kappa) C_0(E, \mathrm{id})$.*

(c) *If $a_\delta > 0$, $\psi \circ \delta/\kappa$ is a nondecreasing function, E has Fatou norm and $P_\psi \in [E_{\kappa,\delta}]$, then $\beta^l(E) < q^l(\psi \circ \delta/\kappa)$.*

PROOF. Since $(P_\psi x)(\delta(t)) \leqslant b_\delta P_{\psi\circ\delta}(x \circ \delta)(t)$ for almost all $t \in I$ it is sufficient to prove (a) and (b) for $\delta(t) = t$.

(a) See [**15**, p. 408].

(b) (See Pavlov [**19**] for P_1.) We have

$$\begin{aligned}
(\kappa(t)P_\psi x(t))^* &\leqslant (\kappa(t)P_\psi x(t))^{**} = t^{-1} \sup_{me=t} \int_e |\kappa(u)P_\psi x(u)|\, du \\
&\leqslant t^{-1}\int_0^1 \left[\sup_{me=t} \int_e |x(us)|\, \kappa(u) \frac{\psi(us)}{\psi(u)}\, du \right] ds/s \\
&= \int_0^1 [x(ts)\kappa(t)\psi(ts)/\psi(t)]^{**}\, ds/s \\
&\leqslant \int_0^1 [\sigma_{1/s}(x\kappa)(t)]^{**} M^l(s, \psi/\kappa)\, ds/s \\
&= \int_0^1 \sigma_{1/s}[(x\kappa)^{**}(t)]\, M^l(s, \psi/\kappa)\, ds/s.
\end{aligned}$$

From Lemma 1 follows

$$\|\kappa(t)P_\psi x(t)\|_E \leqslant \int_0^1 h(s^{-1}, E) M^l(s, \psi/\kappa)\, ds/s\, \|(x\kappa)^{**}\|_E.$$

Since $\beta^l(E) < 1$, by Shimogaki's theorem (see [**10**, p. 187]),

$$\|(x\kappa)^{**}\|_E \leqslant \int_0^1 h(s^{-1}, E)\, ds\, \|x\kappa\|_E.$$

Hence $P_\psi \in [E_{\kappa,t}]$ and $\|P_\psi\| \leqslant C_0(E, \psi/\kappa)C_0(E, \mathrm{id})$.

(c) From the inequality $\kappa(t)(P_\psi x)(\delta(t)) \geqslant a_\delta P_{\psi \circ \delta/\kappa}(\kappa x \circ \delta)(t)$ a.e. on I and from Theorem 2 of [**15**] we have our thesis.

As a consequence of Theorem 1 one obtains

COROLLARY 1 (PAVLOV [**18**, THEOREM 2] OR [**10**, p. 194]; [**19**, THEOREM 2]). *Let κ be a positive measurable and submultiplicative function on $\mathbf{R}_+$. If E is a r.i. space on $\mathbf{R}_+$ such that $\beta^\infty(E) < 1 - q^\infty(\kappa)$ and E has Fatou norm or $\beta^\infty(E) < 1$, then $P_1 = S_1 \in [E_{\kappa,t}]$.*

PROPOSITION 2. *Let E, δ and κ be as in Theorem 1. If $b_\delta < \infty$, $\beta^l(E) < p^l(\delta/\kappa)$ and E has Fatou norm or $\beta^l(E) < 1$, then*

$$E^{(*)}_{\kappa,\delta} = E^{(**)}_{\kappa,\delta}. \tag{20}$$

PROOF. This follows immediately from Theorem 1.

THEOREM 2. *Let E be a r.i. space on I and let $\psi\colon I \to \mathbf{R}_+$, $\delta\colon I \to I$ be increasing absolutely continuous functions and $\kappa\colon I \to \mathbf{R}_+$ a measurable function such that $\delta(0^+) = 0$ and $p^l(\psi \circ \delta/\kappa) > 0$.*

(a) *If $b_{\psi \circ \delta} < \infty$, E has Fatou norm and $\beta^l(E) < p^l(\psi \circ \delta/\kappa)$, then $S_\psi \in [E_{\kappa,\delta}]$ and $\|S_\psi\| \leqslant b_{\psi \circ \delta} C_0(E, \psi \circ \delta/\kappa)$.*

(b) *If $b_{\psi \circ \delta} < \infty$ and $\beta^l(E) < \min\{1, p^l(\psi \circ \delta/\kappa)\}$, then $S_\psi \in [E_{\kappa,\delta}]$ and $\|S_\psi\| \leqslant b_{\psi \circ \delta} C_0(E, \psi \circ \delta/\kappa) C_0(E, \mathrm{id})$.*

(b′) *If $\psi \circ \delta$ is a concave function on I and $\beta^l(E) < p^l(\psi \circ \delta)$ then $S_\psi \in [E_{1,\delta}]$ and $\|S_\psi\| \leqslant \int_0^1 h(s^{-1}, E)\, dM^l(s, \psi \circ \delta)$.*

(c) *If $S_\psi \in [E_{1,\delta}]$, then $\beta^l(E) < q^l(\psi \circ \delta)$.*

PROOF. Since $(S_\psi x)(\delta(t)) = S_{\psi \circ \delta}(x \circ \delta)(t)$ a.e. on I, so $S_\psi \in [E_{\kappa,\delta}]$ if and only if $S_{\psi \circ \delta} \in [E_{\kappa,t}]$. Moreover,

$$\|S_\psi\|_{[E_{\kappa,\delta}]} = \|S_{\psi \circ \delta}\|_{[E_{\kappa,t}]}. \tag{21}$$

The thesis follows from the facts:

(a) and (b) from (21), the inequality $S_{\psi \circ \delta} \leqslant b_{\psi \circ \delta} P_{\psi \circ \delta}$ and Theorem 1;

(b′) from the inequality

$$\begin{aligned}(S_{\psi \circ \delta} x)(t) &= [\psi \circ \delta(t)]^{-1} \int_0^1 x(ts)\, d\psi(\delta(ts)) \\ &\leqslant [\psi \circ \delta(t)]^{-1} \int_0^1 x^*(ts)\, d\psi(\delta(ts))\end{aligned}$$

and Lemma 1;

(c) see [**16**, Theorem 4.4] (see also [**14**, Theorem 3.4]).

REMARKS 3. The reader will have no difficulty in formulating the results for the Q_ψ and T_ψ operators.

4. Applying Theorems 1 and 2, and Proposition 1 [for $\kappa(t) = 1$ and $\delta(t) = \tau^{-1}(t)$, τ—from Example 1] we have boundedness of P_ψ, Q_ψ, S_ψ and T_ψ operators in $E(\mu)$ spaces. In particular case,

5. If $0 < a_\tau \leq b_\tau < \infty$, $b_{\psi\circ\tau^{-1}} < \infty$, $0 < p^l(\psi\circ\tau^{-1}) = q^l(\psi\circ\tau^{-1})$ and E has Fatou norm, then

$$P_\psi \in [E(\mu)] \Leftrightarrow S_\psi \in [E(\mu)] \Leftrightarrow \beta^l(E) < p^l(\psi\circ\tau^{-1}),$$
$$Q_\psi \in [E(\mu)] \Leftrightarrow T_\psi \in [E(\mu)] \Leftrightarrow \alpha^l(E) > p^l(\psi\circ\tau^{-1}).$$

COROLLARY 2 (ANDERSEN [**1**, THEOREM 1]). *Let $l = \infty$, E have Fatou norm and $\tau(t) = t^\sigma/\sigma$, $\sigma > 0$. From Remark 5 follows*

$$S_a = aP_a \in [E(\mu)] \Leftrightarrow \beta^\infty(E) < a/\sigma,$$
$$T_a = aQ_a \in [E(\mu)] \Leftrightarrow \alpha^\infty(E) > a/\sigma.$$

COROLLARY 3 (V. N. SEDOV, SEE [**11**]). *Let $0 < l \leq \infty$ and $\tau(t) = \psi(t)$. If $p > 1$ then*

$$\beta^l(L^p) = 1/p < 1 = p^l(\psi\circ\tau^{-1}),$$

and by Theorem 2(b′) *we have*

$$S_\psi \in [L^p(\mu)] \quad \text{and} \quad \|S_\psi\|_{[L^p(\mu)]} \leq p/(p-1).$$

More generally, if E is any r.i. space on I with the upper index $\beta^l(E) < 1$, then by Theorem 2(b′) *we have*

$$S_\psi \in [E(\mu)] \quad \text{and} \quad \|S_\psi\|_{[E(\mu)]} \leq \int_0^1 h(s^{-1}, E)\, ds.$$

COROLLARY 4 (BENNETT [**2**, THEOREM 6.2]—FOR $f(t) = t^\alpha$, $\alpha > 0$). (i) *Let $l = 1$, $1 \leq p \leq \infty$, $\tau(t) = (1 - \log t)^{-1}$ and $\psi(t) = (1 - \log t)^{-1/p} f((1 - \log t)^{-1})$, where $f > 0$,*

$$0 < a \leq uf'(u)/f(u) \leq b < \infty \quad \forall 0 < u \leq 1.$$

Since $\psi(\tau^{-1}(t)) = \psi(\exp(1 - 1/t)) = t^{1/p} f(t)$, then

$$p^1(\psi\circ\tau^{-1}) = 1/p + p^1(f) > 1/p + a > 1/p.$$

By Theorem 2(a) *we have $S_\psi \in [L^p(\mu)]$ and*

$$\|S_\psi\|_{[L^p(\mu)]} \leq (b + 1/p)\int_0^1 M(s, f)\, ds/s = A,$$

i.e.

$$(22)\quad \left\| f\left((1 - \log t)^{-1}\right)^{-1} \int_0^t x(s)\, ds \right\|_{L^p(\nu)} \leq \frac{A}{a + 1/p} \|t(1 - \log t) f\left((1 - \log t)^{-1}\right)^{-1} x(t)\|_{L^p(\nu)}$$

where $L^p(\nu) = L^p((0, 1), dt/[t(1 - \log t)])$.

(ii) *Let* $l = 1$, $1 \leq p \leq \infty$, $\tau(t) = (1 - \log t)^{-p}$ *and*

$$\psi(t) = (1 - \log t)^{-1} f\big((1 - \log t)^{-1}\big)^{-1},$$

where $f > 0$,

$$0 < a \leq uf'(u)/f(u) \leq b < 1 \quad \forall 0 < u \leq 1.$$

Since $\psi(\tau^{-1}(t)) = \psi(\exp(1 - t^{-1/p})) = t^{1/p} f(t^{1/p})^{-1}$, *then*

$$q^1(\psi \circ \tau^{-1}) = \frac{1}{p} - \left(\frac{1}{p}\right) p^1(f) < \frac{1}{p} - \left(\frac{1}{p}\right) a < \frac{1}{p}.$$

Hence we have $T_\psi \in [L^p(\mu)]$ *and*

$$\|T_\psi\|_{[L^p(\mu)]} \leq \frac{1-a}{p} \int_1^\infty M(s^{-1/p}, f)\, ds/s = B,$$

i.e.

$$\left\| f\big((1 - \log t)^{-1}\big) \int_t^1 x(s)\, ds \right\|_{L^p(\nu)} \tag{23}$$

$$\leq \frac{B}{1-b} \| t(1 - \log t) f\big((1 - \log t)^{-1}\big) x(t) \|_{L^p(\nu)}$$

REMARK 6. Heinig [**9**] proved the boundedness of Hardy $P_1 = S_1$ operator in weighted spaces $L^p((0, l);\ w(x)x^{p-r-1}\,dx)$, where w is a nonnegative, nonincreasing function defined on $(0, \infty)$, $l = \sup\{t > 0\colon\ w(t) > 0\}$, $0 < r < \infty$ and $p \geq 1$. If $\tau'(t) = w(t)t^{p-r-1}$ then, in order to apply Theorem 2, additional assumptions about w, p and r are needed; however, taking $\tau(t) = t$ and manipulating with the function ψ, we obtain Heinig's result (see [**15**, Corollary 3]). I suspect that this result is also obtained in the paper by Sedaev [**20**], which I only know from the review in RŽ Math. 1973, 8 #681.

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