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STRONGLY EXPOSED POINTS IN BOCHNER L^p-SPACES

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ABSTRACT. We give necessary and sufficient conditions for vector-valued L^p-functions to be strongly exposed in terms of their values $(1 < p < \infty)$.

In this paper we give a characterization of the strongly exposed vector-valued L^p-functions in terms of their values $(1 < p < \infty)$. In [**4**] J. A. Johnson has shown that, given a finite positive measure space (Ω, Σ, μ), a Banach space V, an x in $L^p(\mu, V)$ and a g in $L^q(\mu, V')$ (where $1/p + 1/q = 1$), then x is strongly exposed by g if the scalar function $\|x(\cdot)\|$ is strongly exposed by $\|g(\cdot)\|$ and for almost all t with $x(t) \neq 0$ the value $x(t)$ is strongly exposed by $g(t)$. He left the converse as an open question, but gave a kind of supplement in the case that V has RNP. It is not too obvious that the converse should hold. Namely, a similar characterization of the extremal points is valid for separable V (and Borel measures on Polish spaces [**3**]), but not in general [**2**].

We are going to show that the converse *does* hold for Radon measures μ on locally compact spaces, no matter what properties V has (Theorem 2). If V is separable we may even admit arbitrary positive measures μ (Theorem 1). In contrast to the extremal point situation the proof is rather simple.

In this manner we shall have a characterization of strong exposure as a relation between elements of $L^p(\mu, V)$ and $L^q(\mu, V')$; however, this is not yet a characterization of strongly exposed points. We shall give such a characterization under additional assumptions concerning V (Theorems 3 and 4).

Recall that an element x of a normed space X is said to be strongly exposed by an element φ of the dual X' if

(i) $\varphi x = \|\varphi\| \cdot \|x\| \neq 0$, and

(ii) each sequence (x_n) in the ball with radius $\|x\|$, such that φx_n converges to φx, converges to x in norm.

For arbitrary functions $x\colon \Omega \to V$ and $g\colon \Omega \to V'$ let us denote the functions $t \mapsto \|x(t)\|$, $\|g(t)\|$ and $g(t)x(t)$ by $|x|$, $|g|$ and $\langle x, g\rangle$, respectively. χ_A is the characteristic function of the subset A of Ω, and $\mathbf{v}$ is the constant function with value v. $B(v, \varepsilon)$ denotes the closed ball with center v and radius ε.

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Before stating the theorems we make a few observations. Assume $g \in L^q(\mu, V')$ strongly exposes $x \in L^p(\mu, V)$. Then from

$$\|g\| \cdot \|x\| = gx = \int \langle x, g \rangle \, d\mu \leqslant \int |g| \cdot |x| \, d\mu \leqslant \|g\| \cdot \|x\|,$$

we deduce $\langle x, g \rangle = |g| \cdot |x|$ a.e. and, consequently, that $\chi_A g$ strongly exposes $\chi_A x$ for all measurable A with $\chi_A x \neq 0$. Thus, in order to prove the theorems below, knowing that x and g vanish outside a suitable σ-finite set, we may assume without loss of generality that μ is finite. For the same reason we may assume that $|x|$ and $|g|$ are strictly positive. We may also assume w.l.o.g. that the scalars are real, since for complex scalars φ strongly exposes x if and only if the real part $\mathrm{Re} \circ \varphi$ of φ strongly exposes x in the underlying real space.

THEOREM 1. *Let (Ω, Σ, μ) be a positive measure space, V a separable Banach space, $1 < p < \infty$, $x \in L^p(\mu, V)$ and $g \in L^q(\mu, V')$. Then g strongly exposes x if and only if $|g|$ strongly exposes $|x|$ (i.e. $|g|/\|g\| = (|x|/\|x\|)^{p-1}$) and for almost all $t \in \Omega$, $g(t)$ strongly exposes $x(t)$ or $g(t) = 0 = x(t)$.*

PROOF. The "if" part is Theorem 1 in [4], where the finiteness of μ is an unnecessary restriction. Now let g strongly expose x. From $\int |g| \cdot |x| \, d\mu = \|g\| \cdot \|x\|$ and the uniform convexity of $L^p(\mu)$, it is clear that $|g|$ strongly exposes $|x|$. By the preceding remarks we can assume that $|x|$ and $|g|$ are strictly positive, μ is finite and the scalars are real. For $t \in \Omega$ and $n \in \mathbf{N}$ define the slices

$$S(t, n) := \{v \in V \mid \|v\| < |x|(t), g(t)v > (1 - 1/n) \cdot |g|(t) \cdot |x|(t)\},$$
$$d(t, n) := \min\{\varepsilon > 0 \mid S(t, n) \subset B(x(t), \varepsilon)\},$$

and

$$e(t) := \inf\{d(t, n) \mid n \in \mathbf{N}\}.$$

Observe that $g(t)$ strongly exposes $x(t)$ if and only if $e(t) = 0$.

First we want to show that e is a measurable function. By the definition of e it is sufficient to show that the functions $d(\cdot, n)$ are measurable, i.e. the sets $\{t \mid d(t, n) > \delta\}$ are measurable for all $\delta > 0$. Fix such a δ. For $v \in V$ define $A_{n,v} := \{t \mid \|v\| < |x|(t), g(t)v > (1 - 1/n) \cdot |g|(t) \cdot |x|(t), \|x(t) - v\| > \delta\}$. $A_{n,v}$ is measurable since all the functions involved are measurable (w.l.o.g. μ is a complete measure). Now let D be a countable dense subset of V. Observe that $d(t, n) > \delta$ if and only if $t \in A_{n,v}$ for a suitable $v \in D$. Thus we conclude that $\{t \mid d(t, n) > \delta\}$ is measurable as a countable union of measurable sets.

It remains to show that $\{t \mid e(t) > 0\}$ has measure zero. Assume the contrary. Then there is a $\delta > 0$ such that $A := \{t \mid e(t) > \delta\}$ has positive measure. We want to construct a sequence (y_n) in $L^p(\mu, V)$ such that $|y_n - x| \geqslant \delta$ on A, $|y_n| \leqslant |x|$ and $\langle y_n, g \rangle \geqslant (1 - 1/n) \cdot |g| \cdot |x|$ a.e. Then $\|y_n\| \leqslant \|x\|$, $\|y_n - x\|^p \geqslant \mu(A) \cdot \delta^p$ and $gy_n \geqslant (1 - 1/n) \cdot \|g\| \cdot \|x\|$, which means that g does not strongly expose x, a contradiction. To this end let $n \in \mathbf{N}$ and define the sets $A_{n,v}$ as above. Then $A \subset \{t \mid d(t, n) > \delta\} = \bigcup_{v \in D} A_{n,v}$. Hence $A = \dot{\bigcup}_{m=1}^{\infty} B_m$, where each B_m is a measurable set contained in some A_{n,v_m}, $v_m \in D$. Obviously $y_n := \chi_{\Omega \setminus A} x + \sum_{m=1}^{\infty} v_m \cdot \chi_{B_m}$ has the desired properties. $\square$

THEOREM 2. *Let μ be a Radon measure on a locally compact space Ω, V any Banach space, $1 < p < \infty$, $x \in L^p(\mu, V)$ and $g \in L^q(\mu, V')$. Then g strongly exposes x if and only if $|g|$ strongly exposes $|x|$ and for almost all $t \in \Omega$, $g(t)$ strongly exposes $x(t)$ or $g(t) = 0 = x(t)$.*

PROOF. We proceed as in the proof of Theorem 1; we have to show that $\{t \mid e(t) > 0\}$ is a null set. By Lusin's theorem the restrictions of x and g to suitable compact subsets, whose complements have arbitrarily small measures, are continuous. Thus we may assume w.l.o.g. that x and g are continuous on Ω. Consequently the sets $A_{n,v}$ are open, and so is their union $\bigcup_{v \in V} A_{n,v} = \{t \mid d(t, n) > \delta\}$. This shows the measurability of e.

In order to verify that $\{t \mid e(t) > 0\}$ is a null set, replace the set A in the proof of Theorem 1 by a compact subset with positive measure ($A \subset \{t \mid e(t) > \delta\}$, A compact, $\mu(A) > 0$) and proceed as above. Then by its compactness A is contained in a finite union of sets $A_{n,v}$, and the functions y_n defined analogously form the desired sequence. □

Since the dual of $L^p(\mu, V)$ is $L^q(\mu, V')$ if V' has RNP, the following is an immediate corollary from the preceding theorems.

THEOREM 3. *Let V' have RNP. Assume that μ is a Radon measure or V is separable. Then for each strongly exposed $x \in L^p(\mu, V)$ almost all values $x(t)$ are strongly exposed or zero.*

As mentioned before this is not yet a *characterization* of strongly exposed points in $L^p(\mu, V)$. Namely, given an $x \in L^p(\mu, V)$ such that almost all $x(t)$ are strongly exposed by some $g(t) \in V'$, we do not know whether the $g(t)$'s fit together in a measurable way. We do, however, if V is smooth.

THEOREM 4. *Let V be smooth, μ arbitrary. Then each $x \in L^p(\mu, V)$ with $x(t)$ strongly exposed or zero a.e. is strongly exposed.*

PROOF. W.l.o.g. $\|x\| = 1$. We may choose an $A \in \Sigma$ s.t. $\chi_A x = 0$ and, for all $t \notin A$, $x(t)$ is strongly exposed by some norm 1 functional $g_0(t)$, and a sequence of simple functions x_n, vanishing on A and taking only strongly exposed values outside A (namely, certain $x(t)$'s), such that $x_n(t) \underset{n}{\to} x(t)$ everywhere. Put $g_0(t) := 0$ for $t \in A$. Since the support mapping $v \mapsto$ norm 1 functional supporting the unit ball in $v/\|v\|$ is norm-$\sigma(V', V)$-continuous on the unit sphere [**1**, p. 22], hence everywhere on $V \setminus \{0\}$, g_0 is the weak-$*$ limit of a sequence of simple functions, hence weak-$*$ measurable. From this it is easy to see that $\langle y, g_0 \rangle$ is measurable for all $y \in L^p(\mu, V)$ and that $\varphi y := \int \langle y, |x|^{p-1} g_0 \rangle \, d\mu$ defines a linear functional on $L^p(\mu, V)$ with $\varphi x = 1$ and $\|\varphi\| = 1$. Although $g := |x|^{p-1} g_0$ need not be Bochner measurable, $|g|$ is in $L^q(\mu)$ and the proof of [**4**, Theorem 1] shows that φ strongly exposes x. □

Added in proof. We can dispose of the RNP requirement in Theorem 3.

THEOREM 3'. *Assume that μ is a Radon measure or V is separable. Then for each strongly exposed $x \in L^p(\mu, V)$ almost all values $x(t)$ are strongly exposed or zero.*

This is a consequence of the facts that 1. any functional φ on $L^p(\mu, V)$ may be regarded as a weak-$*$ measurable function $g\colon \Omega \to V'$ such that the upper integral $\int^* |g|^q\, d\mu$ equals $\|\varphi\|^q$ and $\varphi y = \int \langle y, g \rangle\, d\mu$ [**5**, p. 97], and 2. that Theorems 1 and 2 are valid also for these g (where the measurability of $|g|$ is implicit in "$|g|$ strongly exposes $|x|$"). In order to verify the second fact recall that the proof of [**4**, Theorem 1] shows the sufficiency. If in the paragraph preceding Theorem 1 we replace $|g|$ by a measurable function $f \geqslant |g|$ such that $|g|^q$ and f^q have the same (upper) integral, the arguments of this paragraph prove that $\langle x, g \rangle = |g| \cdot |x| = f \cdot |x|$ a.e. and $f = 0$ a.e. on $\{t \mid |x|(t) = 0\}$. Consequently $|g|$ is measurable. But then (again assuming w.l.o.g. that $|x|$ and $|g|$ are strictly positive) the proofs of Theorems 1 and 2 work, since we only needed that the functions $\langle \mathbf{v}, g \rangle$ $(v \in V)$ and x and $|g| \cdot |x|$ are measurable. $\square$

REMARK. Although each strongly exposing g has a measurable norm function $|g|$, it is not true that g itself is measurable: as $L^p(\mu, l^1)$ has RNP, the strongly exposing functionals are dense in its dual which contains $L^q(\mu, l^\infty)$ as a proper closed subspace because l^∞ lacks RNP (μ not purely atomic).

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