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AN ANALOGUE OF THE SCHWARZ LEMMA FOR BOUNDED SYMMETRIC DOMAINS[1]

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ABSTRACT. We determine the best possible estimate for the determinant of the Jacobian of a holomorphic mapping of a bounded symmetric domain into a ball.

1. This note is concerned with the following analogue of the classical Schwarz lemma. Suppose $D = \prod_\mu D_\mu$ is a bounded symmetric domain in $\mathbf{C}^n$ with irreducible components D_μ, realized as a circular starlike bounded domain with center 0, in accordance with Harish-Chandra's imbedding (here and in the sequel the reference for the theory of bounded symmetric domains is [**5**]). Let $F: D \to B_n$ be a holomorphic mapping of D into the unit ball of $\mathbf{C}^n$. Let $J(F)$ denote the Jacobian matrix of F. To estimate $\det(J(F)(z))$ we can suppose $F(0) = 0$ and evaluate the Jacobian at the origin (we write $J(F)(0) = J(F)$). Let l_μ and n_μ denote the rank and the dimension of D_μ respectively. We prove that

$$|\det(J(F))| \leq n^{-n/2} \cdot \prod_\mu (n_\mu/l_\mu)^{n_\mu/2},$$

and that this estimate cannot be improved.

The above inequality was proved by Carathéodory [**2**] for the polydisc and by Kubota [**7**] for the classical Cartan domains. Related results may be found in [**3**] under more general hypotheses, but in our case these results do not give sharp estimates. Let us refer also to Korányi [**4**] for a different, and more classical, extension of the Schwarz lemma to bounded symmetric domains.

2. Let $D_\mu = G_\mu/K_\mu$, where G_μ is the connected component of the group of holomorphic automorphisms of D_μ and K_μ is the subgroup of G_μ which leaves 0 fixed (K_μ is a connected compact group of unitary transformations). Let $\mathfrak{g}_\mu^{\mathbf{C}}$ and $\mathfrak{k}_\mu^{\mathbf{C}}$ be the complexifications of the Lie algebras of G_μ and K_μ respectively. Under the symmetry σ_μ of $\mathfrak{g}_\mu$ we have the decomposition $\mathfrak{g}_\mu = \mathfrak{k}_\mu + \mathfrak{p}_\mu$ into eigenspaces of σ_μ for the eigenvalues $+1$ and -1 respectively. We choose a Cartan subalgebra $\mathfrak{h}_\mu$ in $\mathfrak{k}_\mu$: then $\mathfrak{h}_\mu^{\mathbf{C}}$ is a Cartan subalgebra in $\mathfrak{g}_\mu^{\mathbf{C}}$. To every noncompact root α_μ (a root of $\mathfrak{g}_\mu^{\mathbf{C}}$ which is not a root of $\mathfrak{k}_\mu^{\mathbf{C}}$) we associate in the standard way the element E_α^μ of $\mathfrak{g}_\mu^{\mathbf{C}}$. The canonical realization of D_μ is in the complex vector space $\mathfrak{p}_\mu^-$ which is the subalgebra of $\mathfrak{g}_\mu^{\mathbf{C}}$ spanned by the $E_{-\alpha}$ (α positive noncompact). Let Δ_μ denote the Harish-Chandra

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system of strongly orthogonal noncompact positive roots. It is known that $\mathfrak{a}_\mu = \sum_{\alpha_\mu \in \Delta_\mu} \mathbf{R}(E^\mu_\alpha + E^\mu_{-\alpha})$ is a maximal abelian subalgebra contained in $\mathfrak{p}_\mu$. Let $A_\mu = \exp \mathfrak{a}_\mu$. Since $G_\mu = K_\mu A_\mu K_\mu$ we have $D_\mu = K_\mu A_\mu \cdot 0$, and the orbit $A_\mu \cdot 0$ is a unit (l_μ-dimensional) cube around 0.

We shall use the above notation with μ missed when we shall refer to D.

3. We state the theorem.

THEOREM. *Let $D = \prod_\mu D_\mu$ be a bounded symmetric domain in $\mathbf{C}^n$, with irreducible components D_μ, in the standard realization. Let F be a holomorphic mapping of D into the unit ball B_n in $\mathbf{C}^n$ such that $F(0) = 0$. Then (writing $J(F)(0) = J(F)$)*

$$|\det(J(F))| \leqslant n^{-n/2} \cdot \prod_\mu (n_\mu/l_\mu)^{n_\mu}$$

(n_μ and l_μ are the dimension and the rank of D_μ respectively). Moreover, there exists a mapping $\tilde{F}\colon D \to B_n$ for which the equality holds.

PROOF. Fix a vertex E^μ of the unit cube $A_\mu \cdot 0$ contained in D_μ. Let $S_\mu = K_\mu \cdot E^\mu$ denote the Silov boundary of D_μ (K_μ acts transitively on S_μ). We put $E^\mu_1 = E^\mu \cdot l_\mu^{-1/2}$ and we choose an orthonormal basis $\{E^\mu_i\}_{i=1}^{n_\mu}$ in D_μ. Let $z_1^{(\mu)}, \ldots, z_{n_\mu}^{(\mu)}$ be the coordinate functions associated to this basis.

Now let $z_i^{(\mu)}$ and $z_j^{(\gamma)}$ be two coordinate functions on the domains D_μ and D_γ respectively. We denote by S the Silov boundary of D and we have (for $\mu \neq \gamma$)

$$\int_S z_i^{(\mu)} \overline{z_j^{(\gamma)}} = \int_{S_\mu} z_i^{(\mu)} \cdot \int_{S_\gamma} \overline{z_j^{(\gamma)}} = 0 \qquad (\mu \neq \gamma). \tag{1}$$

For $\mu = \gamma$ we write z_i in place of $z_i^{(\mu)}$ and we have

$$\int_S z_i \overline{z_j} = \int_{S_\mu} z_i \overline{z_j} = \int_{K_\mu} z_i(k \cdot E^\mu) \cdot \overline{z_j(k \cdot E^\mu)}\, dk. \tag{2}$$

But

$$\begin{aligned} z_i(k \cdot E^\mu) &= (k \cdot E^\mu, E^\mu_i) = l_\mu^{1/2} \cdot (k \cdot E^\mu_1, E^\mu_i) \\ &= l_\mu^{1/2} \cdot \mathrm{Ad}_{\mathfrak{p}^-_\mu}(k)_{1,i} \end{aligned} \tag{3}$$

where $\mathrm{Ad}_{\mathfrak{p}^-_\mu}(k)_{1,i}$ is the $(1, i)$-coefficient of the adjoint representation of K_μ acting on the complex space $\mathfrak{p}^-_\mu$ in which D_μ is realized. Since D_μ is irreducible, the n_μ-dimensional unitary representation $\mathrm{Ad}_{\mathfrak{p}^-_\mu}$ is irreducible. Then (1), (2), (3) and Schur's lemma give

$$\begin{aligned} \int_S z_i^{(\mu)} \overline{z_j^{(\gamma)}} &= \delta_{\mu,\gamma} \cdot l_\mu \int_{K_\mu} \mathrm{Ad}_{\mathfrak{p}^-_\mu}(k)_{1,i} \cdot \overline{\mathrm{Ad}_{\mathfrak{p}^-_\mu}(k)_{1,j}}\, dk \\ &= \delta_{\mu,\gamma} \cdot \delta_{i,j} \cdot l_\mu/n_\mu \quad \text{(Kronecker's } \delta\text{)}. \end{aligned} \tag{4}$$

We now observe that if P_σ and P_ν are homogeneous polynomials on $\mathfrak{p}^-$ with degree σ and ν respectively, then

$$\int_S P_\sigma \overline{P_\nu} = 0 \quad (\text{if } \sigma \neq \nu). \tag{5}$$

To prove (5) we can use the so-called Bochner's Trick (see [**1**, **6**]). First, we recall that K contains all the elements of the form $e^{i\theta}I$, where θ is any real number and I is the identity operator. Then, by homogeneity,

$$\int_S P_\sigma \overline{P_\nu} = \frac{1}{2\pi}\int_S \int_0^{2\pi} P_\sigma(e^{i\theta}z) \cdot \overline{P_\nu(e^{i\theta}z)}\, ds\, d\theta$$

$$= \frac{1}{2\pi}\int_S \int_0^{2\pi} e^{i(\sigma-\nu)\theta} \cdot P_\sigma(z) \cdot \overline{P_\nu(z)}\, ds\, d\theta = 0 \quad (\text{if } \sigma \neq \nu).$$

Now, let F be as in the statement of the Theorem. We write $F = (f_1,\dots,f_n)$ and $f_i(z_1,\dots,z_n) = a^i_1 z_1 + \cdots + a^i_n z_n + (\text{higher terms})$, $i = 1,\dots,n$. The almost everywhere defined boundary values of the bounded holomorphic mapping F will also be denoted by F. Thus

$$1 \geqslant \int_S |F|^2 = \sum_{i=1}^n \int_S |f_i|^2.$$

We develop each f_i in homogeneous polynomials and we use the orthogonality relations (4) and (5) to obtain

$$(6) \qquad 1 \geqslant \sum_{i=1}^n \int_S |a^i_1 z_1 + \cdots + a^i_n z_n + (\text{higher terms})|^2 \geqslant \sum_{i=1}^n (l_{\mu_i}/n_{\mu_i}) \cdot |a^i_j|^2$$

where D_{μ_i} is the domain on which z_i is defined.

Now let H be the $n \times n$ matrix with coefficients $h^i_j = (l_{\mu_i}/n_{\mu_i})^{1/2} \cdot a^i_j$. Then

$$(7) \qquad \operatorname{trace}(HH^*) = \sum_{i,j=1}^n (l_{\mu_i}/n_{\mu_i}) \cdot |a^i_j|^2.$$

Here HH^* is a positive definite Hermitian matrix. Hence, it is unitarily equivalent to a diagonal matrix V with strictly positive entries on the diagonal. Then

$$(8) \qquad \operatorname{Trace}(HH^*) = \operatorname{Trace} V \geqslant n(\det(V))^{1/n} = n(\det HH^*)^{1/n}.$$

Finally, since $\det(HH^*) = \Pi_\mu (l_\mu/n_\mu)^{n_\mu} \cdot |\det(a^i_j)|^{1/2}$, we get from (6), (7) and (8):

$$(9) \qquad |\det J(F)| = |\det(a^i_j)| = \prod_\mu (n_\mu/l_\mu)^{n_\mu/2} \cdot |\det(HH^*)|^{1/2}$$

$$\leqslant n^{-n/2} \cdot \prod_\mu (n_\mu/l_\mu)^{n_\mu/2}.$$

We conclude the proof by getting a mapping $\tilde{F}$ for which the equality holds in (9). Let $\tilde{F}: D \to \mathbf{C}^n$ be such that $F = (\tilde{f}_1,\dots,\tilde{f}_n)$, where, for $i = 1,\dots,n$,

$$\tilde{f}_i(z_1,\dots,z_n) = n^{-1/2} \cdot (n_\mu/l_\mu)^{1/2} \cdot z_i$$

where D_μ is the irreducible domain on which the coordinate function z_i was defined. Observe that $D_\mu = K_\mu A_\mu \cdot 0$ is contained in an n_μ-dimensional ball of radius $l_\mu^{-1/2}$. Hence $\tilde{F}|_{D_\mu}(D_\mu)$ is contained in a ball of radius $(n_\mu/n)^{1/2}$. Hence $\tilde{F}(D)$ is contained in a ball of radius $(n^{-1} \cdot \Sigma_\mu n_\mu)^{1/2} = 1$, i.e. $\tilde{F}(D) \subseteq B_n$. Now, clearly,

$$|\det(J(\tilde{F}))| = n^{-n/2} \cdot \prod_\mu (n_\mu/l_\mu)^{n_\mu/2}$$

and the proof is complete.

REMARK. The orthogonality relation (5) becomes unnecessary if we show that F may be chosen linear without loss of generality. This is not hard to prove. Indeed a standard application of the classical Schwarz lemma shows that the following holds. Let

$$F = \big(a_1^i z_1 + \cdots + a_n^1 z_n + \text{(higher terms)} \\ + \cdots + a_1^n z_1 + \cdots + a_n^n z_n + \text{(higher terms)}\big)$$

be a mapping from the bounded symmetric domain D into B_n; then also the linear map

$$F_{\text{lin}} = \big(a_1^1 z_1 + \cdots + a_n^1 z_n, \ldots, a_1^n z_1 + \cdots + a_n^n z_n\big)$$

sends D into B_n.

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ADDED IN PROOF. A similar result has been independently obtained by Y. Kubota in a paper to appear in Bull. London Math. Soc.

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