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NONEXISTENCE OF INVARIANT MEASURES

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ABSTRACT. Let G be a group acting on a set X. Suppose that for some positive integer r, G contains a free group F of rank $> r$ and the intersection of any stabilizer with F has rank $\leqslant r$. A graph theoretic approach is used to show that there is no invariant measure on X.

1. Introduction. Let G be a group acting on a set X. By an *invariant measure* for this action we will mean a finitely additive, nonnegative measure μ, defined on all subsets of X, such that $\mu(X) = 1$ and $\mu(gA) = \mu(A)$, for all $g \in G$ and $A \subseteq X$. For any $x \in X$ let G_x denote the stabilizer subgroup at x. That is,

$$G_x = \{g \in G: gx = x\}.$$

Consider the question of whether or not an invariant measure exists. If G is amenable (i.e. an invariant measure exists when $X = G$ and the action is by multiplication), it is not difficult to show that an invariant measure exists for all actions (see Greenleaf [**5**]). Our question then is of interest only for nonamenable groups. In particular we consider groups which contain a nonabelian free group. These are always nonamenable, and until recently were the only known examples of such.

In [**10**, Proposition 3.5] Rosenblatt shows that if an invariant measure exists, and G_x is amenable for all x, then G is amenable. It follows immediately that if G contains a free group F of rank greater than 1 and if for all x, $G_x \cap F$ is of rank $\leqslant 1$, therefore abelian, then no invariant measure exists. This same result was obtained independently by Akemann, using quite different methods (see [**1**, Proposition 4.7]). The purpose of this note is to prove the following generalization of the above result.

THEOREM. *Let G act on X. Suppose that for some positive integer r, G contains a free group F with rank $\geqslant r + 1$ (possibly infinite) and that $G_x \cap F$ has rank $\leqslant r$ for all $x \in X$. Then no invariant measure exists.*

Our techniques seem to differ somewhat from the usual ones used on problems of this type. We use a graph theoretic approach. Before giving the proof we will review briefly some background concepts.

2. Graph theory concepts. We will basically follow the terminology of Imrich [**7**]. See also Berge [**2**, Chapter 2]. Let $\Gamma = (V, E)$ be a connected directed graph where V is the set of vertices and E the set of edges. To each $e \in E$ we associate the inverse

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edge e^{-1} for which we reverse the initial and terminal vertices of e. The inverse edges are considered distinct from the set E. Fix a vertex p. Let C_p be the set of all walks (i.e. a finite sequence of adjacent edges or inverse edges) beginning and ending at p. Under a natural equivalence relation and operation, the equivalence classes of C_p form a group $\pi_1(p)$, called the *fundamental group* of the graph at p. This turns out to be a free group and its rank $\nu(\Gamma)$, independent of p by connectivity, is called the *cyclomatic number* of the graph Γ.

For another definition of cyclomatic number, consider the map T from C_p into the vector space of all real valued functions on E, defined as follows. For any $\omega \in C_p$ and $e \in E$, let

$$[T(\omega)](e) = (\text{number of occurrences in } \omega \text{ of } e) - (\text{number of occurrences in } \omega \text{ of } e^{-1}).$$

The subspace Z generated by the image of T, which is independent of p by connectivity, is called the *cycle space* of Γ and its dimension is $\nu(\Gamma)$. From this formulation it is immediate that for a connected subgraph $\Gamma' = (V', E')$,

$$\nu(\Gamma') \leqslant \nu(\Gamma) \tag{1}$$

since the cycle space of the subgraph can be identified with a subspace of Z, namely those functions in Z which vanish on the edges not in E'.

Let $|\ |$ denote cardinality. It is well known that when V and E are finite

$$\nu(\Gamma) = |E| - |V| + 1. \tag{2}$$

To any action of a group G on X and a subset S of G we associate a graph Γ as follows. The vertex set is X and the edge set is $(S \times X)$. The edge (g, x) has initial vertex x and terminal vertex gx. (This is known as the *Caley graph* in the case that $X = G$ and the action is by multiplication.) Let F be the group generated by S. Then the orbits for the restriction of the action to F correspond to the connected components of Γ. Fix a vertex p and let Γ_p be the component containing p. The map which assigns g to the edge $e = (g, x)$, and g^{-1} to its inverse, induces a homomorphism from $\pi_1(p)$ onto $G_p \cap F$. This will be 1-1 precisely when S forms a free set of generators for F. So in such a case we have

$$\operatorname{rank}(G_p \cap F) = \nu(\Gamma_p). \tag{3}$$

3. Conclusion.

PROOF OF THE THEOREM. We first reduce to the case where F is finitely generated. Suppose that A is an infinite set of free generators for F. Choose any finite $A_0 \subseteq A$ of cardinality $> r$ and let F_0 be the subgroup generated by A_0. Using, for example, the Kurosch subgroup theorem [**6**, Theorem 17.3.1] or [**8**, p. 117, Exercise 32], we see that for any subgroup H of F, $H \cap F_0$ is a factor in some free product decomposition of H and so $\operatorname{rank}(H \cap F_0) \leqslant \operatorname{rank}(H)$. We can therefore replace F by the finitely generated group F_0. Accordingly let $S = \{g_1, g_2, \ldots, g_t\}$ be a set of free generators for F where t is finite and $> r$. We first want to show that for any finite nonempty $Y \subseteq X$,

$$\sum_{i=1}^{t} |g_i Y \cap Y| \leqslant (t-1)|Y|. \tag{4}$$

To do so we form the graph corresponding to S as indicated above and let $\Gamma' = (Y, E')$ be the full subgraph on Y (i.e., E' consists of all edges with initial and terminal vertices in Y). We can assume that Γ' and Γ are connected, since if (4) holds on each component, it will clearly hold globally. From (1), (2), (3) and our hypothesis on rank,

$$(5) \qquad \begin{aligned} |E'| &= \nu(\Gamma') - 1 + |Y| \leqslant \nu(\Gamma) - 1 + |Y| \leqslant (r-1) + |Y| \\ &\leqslant (r-1)|Y| + |Y| = r|Y| \leqslant (t-1)|Y|. \end{aligned}$$

This establishes (4), as the number of edges in E' with g_i as first coordinate $= |Y \cap g_i^{-1}Y| = |g_iY \cap Y|$ which shows that the left-hand side of (4) is just $|E'|$.

We now appeal to the well-known Følner condition. See Rosenblatt **[9]** for a very general treatment. This condition says that an invariant measure exists iff given any finite nonempty set $S \subseteq G$ and any $\varepsilon > 0$ there is a finite nonempty $Y \subseteq X$ such that

$$(6) \qquad |gY \cap Y| \geqslant (1 - \varepsilon)|Y|$$

for all $g \in S$.

Given S as above, $\varepsilon < t^{-1}$, and any finite nonempty Y, (6) cannot hold for all $g \in S$ as this would contradict (4). Hence, no invariant measure exists.

REMARK. The theorem is obviously false for $r = 0$ since F could be of rank 1. An attempt to adapt the proof would break down precisely on the last line of (5), since $(r-1) < 0$.

EXAMPLE. In the case that F has infinite rank the theorem is not necessarily true if we simply require that each $G_x \cap F$ be of finite rank. The uniform bound is needed. Consider the following example, which appeared in **[3]**. G is generated by $\{g_1, g_2, g_3, \ldots, h_1, h_2, h_3, \ldots\}$ subject to the relations that h_i and h_j commute for all i, j and that g_i and h_j commute for $i \leqslant j$. Let F be the subgroup generated by $\{g_1, g_2, \ldots\}$, a free group of infinite rank. Let X consist of all nonidentity elements of G and let G act on X by conjugation. For x equal to a product of h_i's and their inverses, with s being the minimum index i which is needed, $G_x \cap F =$ the group generated by $\{g_1, g_2, \ldots, g_s\}$. For all other x, $G_x \cap F$ is trivial. There is however an invariant measure. In other words, G is *inner amenable* in the terminology of **[4]**. This follows from the results of **[3 and 4]** but the most direct way to see this is simply to use the Følner condition. For any finite subset S of G there is a one point S-invariant set, namely $\{h_N\}$ for N sufficiently large. It does follow from our theorem that the measure of a finite union of conjugacy classes will be zero for any invariant measure on X.

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