PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 1, May 1983

IDEALS OF REGULAR OPERATORS ON l^2

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ABSTRACT. Let $\mathcal{L}^r$ be the Banach algebra (and Banach lattice) of all regular operators on l^2, i.e. the algebra of all operators A on l^2 which are given by a matrix (a_{mn}) such that $(|a_{mn}|)$ defines a bounded operator $|A|$. We show that there exists exactly one nontrivial closed subspace of $\mathcal{L}^r$ which is both a lattice-ideal and an algebra-ideal of $\mathcal{L}^r$, namely the space $\mathcal{K}^r = \{A \in \mathcal{L}^r: |A| \text{ is compact}\}$. We also show that every nontrivial ideal in $\mathcal{L}^r$ is included in $\mathcal{K}^r$.

It is well known that the only nontrivial closed (algebra) ideal in the Banach algebra $\mathcal{L}(H)$ of all bounded operators on a separable Hilbert space H is the ideal $\mathcal{K}(H)$ consisting of all compact operators on H and that $\mathcal{K}(H)$ includes every nontrivial ideal. In this note we prove order-theoretic analogues of these results.

To give a precise statement we need some notation. Let $H = l^2$, the Hilbert space of all (real or complex) square-summable sequences, and denote by $\{e_n\}$, $n = 1, 2, \ldots$, the standard basis in l^2. Every bounded operator A on l^2 can be represented by a matrix (a_{mn}) given by $a_{mn} = (Ae_n \mid e_m)$, $m, n = 1, 2, \ldots$, where $(f \mid g)$ denotes the scalar product of $f, g \in l^2$. If the matrix $(|a_{mn}|)$ also defines a bounded operator $|A|$, then A is called *regular*, and $|A|$ is called the *modulus* of A. For example, every operator of finite rank is obviously regular, but there exist compact operators which are not regular [**4**, IV, §1, Example 2]. We denote by $\mathcal{L}^r$ the space of all regular operators on l^2. It is known that $\mathcal{L}^r$ is a subalgebra of $\mathcal{L}(l^2)$ and is a Banach algebra under the *r-norm* $\|A\|_r$ defined as the operator norm of $|A|$, i.e.

$$\|A\|_r = \| \, |A| \, \|$$

where $\|T\|$ denotes the operator norm of T (cf. [**4**, IV, §1]). A bounded operator A on l^2 is called *positive* (we write $A \geqslant 0$) if $(Ae_n \mid e_m) \geqslant 0$ for all natural numbers m, n. With this order relation, $\mathcal{L}^r$ becomes a Banach lattice (cf. [**4**, IV, §1]). A *lattice-ideal* of $\mathcal{L}^r$ is, by definition, a subspace $\mathcal{J}$ of $\mathcal{L}^r$ such that if $A \in \mathcal{J}$, $B \in \mathcal{L}^r$, $|B| \leqslant |A|$, then $B \in \mathcal{J}$. For the purpose of this paper, we say briefly that $\mathcal{J}$ is an *ideal* if it is both an algebra-ideal *and* a lattice-ideal.

A regular operator is called *r-compact* if A can be approximated in the r-norm by operators of finite rank. The space $\mathcal{K}^r$ of all r-compact operators is a closed ideal in $\mathcal{L}^r$. Our main theorem gives a confirmation of the following conjecture by H. H. Schaefer: $\mathcal{K}^r$ is the only closed nontrivial ideal in $\mathcal{L}^r$. Furthermore, we show that every nontrivial ideal is included in $\mathcal{K}^r$.

Received by the editors June 7, 1982.
1980 *Mathematics Subject Classification*. Primary 47D30; Secondary 47B55, 47B05.

For vectors f, $g \in l^2$, we denote by $f \otimes g$ the rank one operator given by

$$(f \otimes g)h = (h \mid f)g \qquad (h \in l^2).$$

The orthogonal projection onto span$\{e_1, \ldots, e_n\}$ will be denoted by P_n (i.e., $P_n = \Sigma_{m=1}^n e_m \otimes e_m$) and will be called an *initial projection*. Note that $P_n \geqslant 0$, $(1 - P_n) \geqslant 0$ for every $n \in N$ (the set of natural numbers).

The following proposition is well known and easy to prove.

PROPOSITION 1. *Let $A \in \mathfrak{L}(l^2)$. Then*
(i) $\|A\| = \sup_{n \in N} \|P_n A P_n\| = \lim_{n \to \infty} \|P_n A P_n\|$;
(ii) *A is compact if and only if* $\lim_{n \to \infty} \|(1 - P_n)A(1 - P_n)\| = 0$.

We give a characterization of r-compact operators (cf. [**4**, p. 293, Corollary 1]).

PROPOSITION 2. (i) *Let $B, C \in \mathfrak{L}^r$, $|B| \leqslant C$. If C is compact, then $B \in \mathfrak{K}^r$.*
(ii) $\mathfrak{K}^r = \{A \in \mathfrak{L}^r : |A| \text{ is compact}\}$.
(iii) *$\mathfrak{K}^r$ is a closed ideal in $\mathfrak{L}^r$.*

PROOF. Let $F_n = P_n B P_n + (1 - P_n) B P_n + P_n B(1 - P_n)$. Then F_n is an operator of finite rank and

$$|B - F_n| = |(1 - P_n)B(1 - P_n)| \leqslant (1 - P_n)|B|(1 - P_n) \leqslant (1 - P_n)C(1 - P_n).$$

It follows from Proposition 1 (ii) that $\lim_{n \to \infty} \|B - F_n\|_r = 0$. Hence $B \in \mathfrak{K}^r$. This proves part (i).

From (i), it follows that $\mathfrak{K}^r$ includes $\{A \in \mathfrak{L}^r : |A| \text{ compact}\}$. To prove the reverse inclusion, first observe that $|F|$ is compact if F is an operator of finite rank (this follows from (i) since if $F = \Sigma_{i=1}^n x_i \otimes y_i$, then $|F| \leqslant \Sigma_{i=1}^n |x_i| \otimes |y_i|$). Now, let $A \in \mathfrak{K}^r$. Then $\lim_{n \to \infty} \|A - F_n\|_r = 0$ for a sequence $\{F_n\}$ of finite rank operators. This implies that $\lim_{n \to \infty} \| |A| - |F_n| \| = 0$. Therefore $|A|$ is a limit of a sequence of compact operators and so is compact.

Part (iii) follows immediately from (i) and (ii). □

THEOREM 1. *Let $\mathfrak{J}$ be a proper ideal in $\mathfrak{L}^r$. Then $\mathfrak{J} \subset \mathfrak{K}^r$.*

PROOF. Let $\mathfrak{J}$ be an ideal in $\mathfrak{L}^r$ which is not included in $\mathfrak{K}^r$. We will show that $\mathfrak{J} = \mathfrak{L}^r$.

There exists a positive operator A in $\mathfrak{J}$ which is not compact. Let P_{n_1} be an initial projection such that $\|P_{n_1} A P_{n_1}\| \geqslant \frac{1}{2}\|A\|$. By induction, we can find a sequence of increasing initial projections $\{P_{n_j}\}$, $j = 1, 2, \ldots$, such that

$$\|(P_{n_{j+1}} - P_{n_j})A(P_{n_{j+1}} - P_{n_j})\| \geqslant \tfrac{1}{2}\|(1 - P_{n_j})A(1 - P_{n_j})\|.$$

Since A is not compact, the norms $\|(1 - P_{n_j})A(1 - P_{n_j})\|$ are bounded below by a positive number δ. Let $B_j = (P_{n_{j+1}} - P_{n_j})A(P_{n_{j+1}} - P_{n_j})$, so $\|B_j\| \geqslant \delta/2$. The operator $B = \sup_{j \in N} B_j$ belongs to $\mathfrak{J}$ since $0 \leqslant B \leqslant A$. Denote by H_j the range of $P_{n_{j+1}} - P_{n_j}$ and let g_j be a positive vector in H_j of norm 1 such that $\|B_j g_j\| \geqslant \delta/2$. Let $R_n = e_1^n \otimes g_n$, where e_1^n is the first basis vector in H_n (i.e., $e_1^k = e_{n_k+1}$), and let $R = \sup_{n \in N} R_n$. Thus R is a positive operator of norm 1. Let $M_n = B_n R_n = e_1^n \otimes B_n g_n$ and let $M = \sup_{n \in N} M_n$, so $M = BR \in \mathfrak{J}$. Finally, let $T = M^*M$. Then $T \in \mathfrak{J}$ and.

$T = \sup_{n \in N} T_n$, where $T_n = M_n^* M_n$. So T_n is represented (with respect to the standard basis) by the diagonal matrix $\Sigma \alpha_n e_1^n \otimes e_1^n$, where $\alpha_n = \| B_n g_n \|^2$. There is a permutation matrix U such that

$$UTU = \operatorname{diag}(\alpha_1, 0, \alpha_2, 0, \alpha_3, \dots).$$

The sequence $\{\alpha_n\}$ is bounded below, therefore the matrix

$$\operatorname{diag}(\alpha_1^{-1}, 0, \alpha_2^{-1}, 0, \alpha_3^{-1}, 0, \dots)$$

defines a bounded positive operator D, and so $F = DUTU = \operatorname{diag}(1, 0, 1, 0, \dots)$ belongs to $\mathcal{J}$. If V is the permutation operator defined by $Ve_{2n-1} = e_{2n}$, $Ve_{2n} = e_{2n-1}$ $(n \in N)$, then $1 = F + VFV \in \mathcal{J}$. Therefore $\mathcal{J} = \mathcal{L}^r$. □

LEMMA. *Every nonzero algebra ideal in $\mathcal{L}^r$ contains the finite rank operators.*

PROOF. (This is the same proof as the well-known proof of the analogous result in $\mathcal{L}(l^2)$; we merely point out that all the operators involved are regular.) Let $\mathcal{J}$ be a nonzero algebra ideal in $\mathcal{L}^r$ and let A be a nonzero operator in $\mathcal{J}$. Pick a vector f_1 such that $\| Af_1 \| = 1$. For two arbitrary vectors f and g, we have $f \otimes g = (Af_1 \otimes g)A(f \otimes f_1)$, and so every operator of rank one belongs to $\mathcal{J}$. Finally, every finite rank operator is a sum of rank one operators and therefore belongs to $\mathcal{J}$. □

As an immediate consequence of Theorem 1 and the Lemma we obtain the main result.

THEOREM 2. *$\mathcal{K}^r$ is the only nontrivial closed ideal in $\mathcal{L}^r$.*

COROLLARY. $\mathcal{K}^r = \{A \in \mathcal{L}^r\colon \lim_{n \to \infty} |A| Re_n = 0$ *for every positive operator* $R\}$.

REMARK 1. There are nontrivial closed lattice-ideals as well as nontrivial closed algebra-ideals in $\mathcal{L}^r$ other than $\mathcal{K}^r$. For example, $\{T \in \mathcal{L}^r\colon \lim_{n,m \to \infty}(Te_n \mid e_m) = 0\}$ is a nontrivial closed lattice-ideal different from $\mathcal{K}^r$. Moreover, there exists a compact operator $A \in \mathcal{L}^r$ such that $|A|$ is not compact [**4**, IV, §1, Example 2]. Hence $\mathcal{K}(l^2) \cap \mathcal{L}^r = \{T \in \mathcal{L}^r\colon T$ is compact$\}$ is a nontrivial closed algebra-ideal in $\mathcal{L}^r$ which contains $\mathcal{K}^r$ properly.

REMARK 2. By Theorem 1, $\mathcal{K}^r$ is the largest ideal in $\mathcal{L}^r$. The smallest ideal $\mathcal{J}_s$ is the one which is generated by the finite rank operators (by the Lemma). Thus it has the form $\mathcal{J}_s = \{T \in \mathcal{L}^r\colon |T| \leq h \otimes h$ for some positive $h \in l^2\}$. This ideal $\mathcal{J}_s$ is properly included in the ideal of Hilbert-Schmidt operators, which, in turn, is properly included in $\mathcal{K}^r$.

REMARK 3. The situation is different if we replace l^2 by $L^2[0, 1]$. For the definition of $\mathcal{L}^r(L^2)$, see [**4**, IV, §1]. The space $\mathcal{K}^r(L^2)$ of all regular operators on L^2 which can be approximated in the r-norm by operators of finite rank is a closed ideal in $\mathcal{L}^r(L^2)$ (this follows from [**4**, p. 293, Corollary 2]); and by the same proof that we used for the Lemma, it is the smallest closed ideal in $\mathcal{L}^r(L^2)$. The space $\mathcal{K}_1 = \{T \in \mathcal{L}^r(L^2)$; $|T|$ is compact$\}$ is obviously closed, and it follows from the Dodds-Fremlin Theorem ([**3**], see also [**1**]) that $\mathcal{K}_1$ is a lattice-ideal. This implies that it is also an algebra-ideal. However, $\mathcal{K}_1$ includes $\mathcal{K}^r(L^2)$ properly (see [**2**, 3.7 in connection with 2.6]). Another nontrivial closed ideal in $\mathcal{L}^r(L^2)$ is $\mathcal{G}$, the space of all regular integral operators (or kernel operators), i.e. operators T of the form

$$Tf(s) = \int k(t, s) f(t)\, dt,$$

where k is a measurable function such that $|k|$ is the kernel of a bounded operator. (The ideal $\mathcal{I}$ is $(L^2 \otimes L^2)^{\perp\perp}$, the band in $\mathcal{L}^r(L^2)$ generated by the finite rank operators [**4**, IV, 9.8].) In the atomic case ($H = l^2$), all regular operators are kernel operators. Here, in the nonatomic case, $\mathcal{I}$ is a proper closed ideal of $\mathcal{L}^r(L^2)$, properly includes $\mathcal{K}^r(L^2)$, and is not comparable to $\mathcal{K}_1$. Furthermore $\mathcal{K}^r(L^2) = \mathcal{K}_1 \cap \mathcal{I}$ [**4**, p. 293, Corollary 2]. In view of this, one nonatomic version of the problem solved in Theorem 2 is the following question: Is $\mathcal{K}^r(L^2)$ the only nontrivial closed ideal in the Banach lattice algebra of kernel operators on L^2?

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