PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 1, May 1983

LOGICS WITH GIVEN CENTERS AND STATE SPACES

PAVEL PTÁK

ABSTRACT. Let B be a Boolean algebra and let K be a compact convex subset of a locally convex topological linear space. Then there exists a logic with the center Boolean isomorphic to B and with the state space affinely homeomorphic to K.

Introduction. In the quantum logic approach to the foundations of quantum mechanics, one identifies the event structure of a system with an orthomodular partially ordered set L (called usually a logic). The set of states is then represented by the set $\mathbb{S}(L)$ of all probability measures on L (see [**4, 7**]). It can be shown that $\mathbb{S}(L)$ is a compact convex set and conversely, it was proved by F. W. Shultz [**6**] that any compact convex subset of a locally convex topological linear space is affinely homeomorphic to $\mathbb{S}(L)$ for a logic L.

The center $C(L)$ of a logic L is the subset of L consisting of all "absolutely compatible" elements. It is known that the center of L is a Boolean algebra (see [**1, 4**]). Obviously, any Boolean algebra is the center of a logic.

Let us now consider the center and the state space simultaneously. The question is if for any Boolean algebra B and any compact convex subset of a LCTLS there exists a logic L such that $C(L) = B$ and $\mathbb{S}(L) = K$. We answer the question in the affirmative. In the construction we use, among other tools, the result of Shultz [**6**] and the technique of R. Greechie [**2**] for constructing orthomodular posets.

Notions. Results. Let us first review the basic definitions and state some auxiliary propositions.

DEFINITION 1. A logic is a set L endowed with a partial ordering $\leqslant$ and a unary operation $'$ such that:

(i) $0, 1 \in L$;
(ii) $a \leqslant b \Rightarrow b' \leqslant a'$ for any $a, b \in L$;
(iii) $(a')' = a$ for any $a \in L$;
(iv) $a \vee a' = 1$ for any $a \in L$;
(v) $\bigvee_{i=1}^{n} a_n$ exists in L whenever $a_n \in L$, $a_n \leqslant a_k'$ for $n \neq k$;
(vi) $b = a \vee (b \wedge a')$ whenever $a, b \in L$, $a \leqslant b$.

In the sequel, we shall reserve the symbol L for logics. One can prove easily that if $a, b \in L$, $a \leqslant b'$ then $a \vee b$, $a \wedge b$ exists in L.

DEFINITION 2. Two elements $a, b \in L$ are called compatible if there are three elements $c, d, e \in L$ such that $c \leqslant d'$, $d \leqslant e'$, $e \leqslant c'$ and $a = c \vee d$, $b = c \vee e$.

Received by the editors May 27, 1982.
1980 *Mathematics Subject Classification*. Primary 81B10, 06C15; Secondary 81C20.

0002-9939/82/0000-1022/$02.00

DEFINITION 3. An element $a \in L$ is called central if a is compatible to any element of L. We denote by $C(L)$ the set of all central elements of L and call $C(L)$ the center of L.

PROPOSITION 1. *The set $C(L)$ with the operations $'$, $\vee$, $\wedge$ inherited from L is a Boolean algebra.*

PROOF. See [**1**, **4**].

DEFINITION 4. Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics. Denote by $\prod_{\alpha \in I} L_\alpha$ the ordinary Cartesian product of the sets L_α and endow the set $\prod_{\alpha \in I} L_\alpha$ with the relation $\leqslant$ and the unary operation $'$ as follows. If $k = \{k_\alpha \mid \alpha \in I\} \in \prod_{\alpha \in I} L_\alpha$ and $h = \{h_\alpha \mid \alpha \in I\} \in \prod_{\alpha \in I} L_\alpha$, then $k \leqslant h$ (resp. $k' = h$) if and only if $k_\alpha \leqslant h_\alpha$ (resp. $k'_\alpha = h_\alpha$) for any $\alpha \in I$. The set $\prod_{\alpha \in I} L_\alpha$ with the above defined $\leqslant$, $'$ is called the product of the collection $\{L_\alpha \mid \alpha \in I\}$.

PROPOSITION 2. *Let $\{L_\alpha \mid \alpha \in I\}$ be a collection of logics. Then $\prod_{\alpha \in I} L_\alpha$ is a logic. If $C(L_\alpha) = \{0, 1\}$ for any $\alpha \in I$ then $C(\prod_{\alpha \in I} L_\alpha)$ is Boolean isomorphic to the Boolean algebra of all subsets of I.*

PROOF. See [**3**, **5**].

DEFINITION 5. A state on a logic L is a mapping s: $L \to \langle 0, 1 \rangle$ such that:

(i) $s(1) = 1$;

(ii) if $a, b \in L$, $a \leqslant b'$ then $s(a \vee b) = s(a) + s(b)$.

Let us denote by $\mathbb{S}(L)$ the set of all states on L. By a result of F. W. Shultz [**6**], any compact convex subset of a LCTLS equals, up to an affine homeomorphism, $\mathbb{S}(L)$ for a logic L (and vice versa, which is obvious).

DEFINITION 6. A logic L is called poor (resp. rigid) if $\mathbb{S}(L) = \varnothing$ (resp. $|\mathbb{S}(L)| = 1$).

It is known (see [**2**, **6**]) that there are (finite) examples of poor and rigid logics.

PROPOSITION 3. *Suppose that L is a poor logic. Put $L_\alpha = L$ for any $\alpha \in I$. Then $\prod_{\alpha \in I} L_\alpha$ is also a poor logic.*

PROOF. Take the mapping f: $L \to \prod_{\alpha \in I} L_\alpha$ such that $f(k) = (k, k, k \ldots)$ for any $k \in L$. If $s \in \mathbb{S}(\prod_{\alpha \in I} L_\alpha)$ then $sf \in \mathbb{S}(L)$.

DEFINITION 7. A mapping f: $L_1 \to L_2$ is called an embedding if f is injective and the following requirements are satisfied.

(i) $f(1) = 1$;

(ii) $f(a') = f(a)'$ for any $a \in L_1$;

(iii) $a \leqslant b$ if and only if $f(a) \leqslant f(b)$;

(iv) if $a \leqslant b'$ then $f(a \vee b) = f(a) \vee f(b)$.

PROPOSITION 4. *Let K be a compact convex subspace of a LCTLS. Take the logic L_1 constructed in* [**6**, *Theorem, p.* 321]. *Thus $\mathbb{S}(L_1) = K$ and moreover, $C(L_1) = \{0, 1\}$ and L_1 can be embedded into a poor logic L_2 with $C(L_2) = \{0, 1\}$.*

PROOF. We must assume here that the reader is well acquainted with the paper [**6**] and with the Greechie representation of logics (see [**2**]). It follows immediately from the construction of [6] that $C(L_1) = \{0, 1\}$ (see e.g. the plan of the construction, p. 321). Further, let us consider the Greechie diagram D_1 of L_1 and the Greechie diagram D of a finite poor logic L exhibited in [2]. Let us choose "points" $d_1 \in D_1$, $d_2 \in D$ such that d_1, d_2 belong to exactly one Boolean block of D_1, D. Form a new Greechie diagram D_2 by taking the union $D_1 \cup D$ and then "identifying" the points d_1, d_2. The diagram D_2 then represents the required logic L_2.

We are now ready to prove our result.

THEOREM. *Let B be a Boolean algebra and let K be a compact convex subset of a LCTLS. Then there exists a logic L such that $C(L)$ is Boolean isomorphic to B and $\mathbb{S}(L)$ is affinely homeomorphic to K.*

PROOF. We may suppose that B is a Boolean algebra of subsets of a set A. Take a logic M such that $C(M) = \{0, 1\}$, $\mathbb{S}(M) = K$ and denote by P the poor extension of M (Proposition 4). Take a point $a \in A$ and write $L_c = P$ if $c \in A - \{a\}$, $L_a = M$. Consider the logic $R = \prod_{d \in A} L_d$. The desired logic L will now be obtained as a sublogic of R. Let us describe the elements of L. An element $r \in R$ belongs to L if and only if there exists a finite partition $\mathcal{P}$ of A, $\mathcal{P} = \{A_i \mid i = 1, 2, \ldots, n\}$ such that $A_i \in B$ for any $i, 1 \leqslant i \leqslant n$, and $r_p = r_q$ as soon as $\{p, q\} \subset A_i$ for an index $i, 1 \leqslant i \leqslant n$. We are to show that L is a logic with $C(L) = B$ and $\mathbb{S}(L) = K$.

Obviously, $1 \in L$ and if $k \in L$ then $k' \in L$. If $k, h \in L$, $k \geqslant h$ then $k = h \vee (k \wedge h')$. Indeed, if $\mathcal{P}, \mathcal{R}$ are partitions corresponding to k, h then $\mathcal{P} \cap \mathcal{R}$ is the partition corresponding to $k' \wedge h$. The rest is obvious. Thus L is a logic.

Further, since $C(L_d) = \{0, 1\}$ for any $d \in A$ then any central element of L must have only the elements 0, 1 for the coordinates. One can check easily that $k = \{k_d \mid d \in A\}$, where any k_d is either 0 or 1, belongs to L if and only if $D = \{d \mid k_d = 1\} \in B$. Consequently, $C(L) = B$.

It remains to prove that $\mathbb{S}(L) = K$. Since $\mathbb{S}(M) = K$, it suffices to show that there is an affine homeomorphism g: $\mathbb{S}(L) \to \mathbb{S}(M)$. Assume that $s \in \mathbb{S}(L)$. For any $m \in M$, denote by k^m the element of L which has m for all its coordinates. Define $g(s)$ such that $g(s)(m) = s(k^m)$. We need to show that g is injective.

Let us suppose that $g(s_1) = g(s_2)$. Take an element $k \in L$ and assume that $\mathcal{P}$ is the partition corresponding to k. Let A_1 be such a set of $\mathcal{P}$ that $a \in A_1$. Denote by $h = \{h_d \mid d \in A\}$ the element of L with $h_d = 0$ if $d \in A_1$, $h_d = 1$ otherwise. It follows from Proposition 3 that $s_1(k \wedge h) = s_2(k \wedge h) = 0$. Since $g(s_1) = g(s_2)$, we see, again applying Proposition 3, that $s_1(k) = s_1(k \wedge h') = s_2(k \wedge h') = s_2(k)$. Hence the mapping g: $\mathbb{S}(L) \to \mathbb{S}(M)$ is injective and the proof is complete.

Let us state explicitly the following special corollary.

COROLLARY. *Given a Boolean algebra B, there exists a poor (resp. rigid) logic L such that $C(L) = B$.*

Let us observe in conclusion that a similar method yields an analogous result for σ-complete logics and σ-additive states. Naturally, the center then cannot be arbitrary since there are Boolean σ-algebras without any σ-additive state.

THEOREM. *Let B be a Boolean σ-algebra of subsets of a set and let K be a compact convex subset of a LCTLS. Then there is a σ-complete logic L such that $C(L)$ is Boolean σ-isomorphic to B and the space of σ-additive states on L is affinely homeomorphic to K.*

REFERENCES

1. J. Brabec and P. Pták, *On compatibility in quantum logics*, Found. Phys. **12** (1982), 207–212.

2. R. Greechie, *Orthomodular lattices admitting no states*, J. Combin. Theory Ser. A **10** (1971), 119–132.

3. S. Gudder, *Uniqueness and existence properties of bounded observables*, Pacific J. Math. **19** (1966), 81–93.

4. ______, *Stochastic methods in quantum mechanics*, Elsevier, North-Holland, Amsterdam, 1979.

5. V. Maňasová and P. Pták, *On states on the product of logics*, Internat. J. Theoret. Phys. **20** (1981), 451–456.

6. F. W. Shultz, *A characterization of state spaces of orthomodular lattices*, J. Combin. Theory Ser. A **17** (1974), 317–328.

7. V. Varadarajan, *Geometry of quantum theory*, vol. 1, Van Nostrand Reinhold, Princeton, N. J., 1968.

DEPARTMENT OF MATHEMATICS, TECHNICAL UNIVERSITY OF PRAGUE, 16627-PRAGUE 6, CZECHOSLOVAKIA