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REGULAR RIEMANNIAN s-MANIFOLDS OF NONCOMPACT TYPE

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ABSTRACT. In this note it is proven that a regular Riemannian s-manifold of noncompact type (see below) cannot be immersed isometrically and equivariantly in R^n.

Our notation, terminology and basic facts will be those of [**3**].

Let $(M, \{S_x\})$ be a connected periodic regular s-manifold which is metrizable, i.e. there is a Riemannian metric g on M which is invariant with respect to the symmetries $\{S_x : x \in M\}$. (Periodicity means that $(M, \{S_x\})$ has finite order [**3**, p. 4].)

We have the group of isometries $I(M, g)$ which is transitive on M [**3**, p. 2].

Contained in $I(M, g)$ we have the group of transvections $G = \mathrm{Tr}(M, \{S_x\})$ [**3**, p. 57] which is generated by the "elementary transvections", i.e. by the isometries $S_x \circ S_y^{-1}$, $x, y \in M$.

About the group G one knows:

(1) G is a connected Lie group [**3**, II 32, I 25].

(2) G is transitive on M [**3**, II 33].

It is known [**3**, IV 24] that under the above conditions $(M, \{S_x\})$ admits two complementary foliations $\mathfrak{F}_1$, $\mathfrak{F}_2$ such that:

(a) $\mathfrak{F}_1$ is invariant and its leaves are regular s-manifolds with solvable group of transvections.

(b) $\mathfrak{F}_2$ is weakly invariant and its leaves are regular s-manifolds with semisimple group of transvections (compare [**2**, p. 208]).

DEFINITION. *We shall say that $(M, \{S_x\})$ is of noncompact type if the foliation $\mathfrak{F}_2$ has noncompact leaves.*

The objective of this note is to prove the following.

THEOREM. *Let $(M, \{S_x\})$ be a connected periodic, regular s-manifold which is metrizable and of noncompact type. Then $(M, \{S_x\})$ admits no isometric equivariant immersion into a finite-dimensional real representation of $G = \mathrm{Tr}(M, \{S_x\})$.*

PROOF. Let us assume the existence of such an isometric immersion (φ, f): $(G, M) \to (I(R^n), R^n)$, where φ is a Lie group monomorphism and f is an isometric

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immersion such that

$$f(gp) = \varphi(g)f(p) \qquad \forall p \in M, g \in G.$$

Let p be a fixed point in M and let $N(p)$ be an open neighborhood of p in M where f is one-to-one.

Let us very briefly recall the definition of the foliation $\mathfrak{F}_2$. Let $\mathfrak{g}$ be the Lie algebra of G.

Let $(V, S, \tilde{R}, \tilde{T})$ be the infinitesimal model of $(M, \{S_x\})$ [**3**, p. 74] and (G, H, σ) the corresponding prime regular homogeneous s-manifold [**3**, p. 53]. With the equivariant version of Levi's decomposition with respect to the compact group K generated by $\mathrm{ad}(H)$ and σ_* (the derivative of σ at e), one may write $\mathfrak{g} = \mathfrak{r} + \mathfrak{s}$. Now one defines $W = \mathfrak{s} \cap V$; then it is "weakly invariant with respect to $(M, \{S_x\})$" [**3**, p. 92] and one defines a submanifold of $(V, S, \tilde{R}, \tilde{T})$, $(W, S', \tilde{R}', \tilde{T}')$, $S' = S \mid W$, $\tilde{R}' = \tilde{R} \mid W$, $\tilde{T}' = \tilde{T} \mid W$.

Then one uses [**3**, IV 7] with "weakly invariant with respect to $(M, \{S_x\})$" instead of invariant, which is true by [**3**, IV 10].

Let us go back to our proof. By using the above Levi decomposition we can obtain two analytic subgroups of G, R for $\mathfrak{r}$ and Q for $\mathfrak{s}$. R is the radical of G and Q is a maximal semisimple analytic subgroup of G and $G = R \cdot Q$ [**4**, p. 244].

It is easy to see from the definition of $\mathfrak{F}_2$ and the action of G [**3**, I 26] that the leaf of the foliation $\mathfrak{F}_2$ which contains the point p is precisely $Q(p)$ (p our chosen point). Then by our hypothesis Q is noncompact.

Let $U(Q) \xrightarrow{\pi} Q$ be the universal covering of Q. Then $U(Q) = A_1 \times \cdots \times A_k \times B_1 \times \cdots \times B_l$, where the A_i's are noncompact simple Lie groups, the B_j's are compact simple Lie groups, and $k \geqslant 1$.

Now for some of the A_i's, say A_1, the orbit $A_1(p)$ is nontrivial. Then it is a connected submanifold of dimension $\geqslant 1$.

Let us put $C_p =$ Connected component of $A_1(p) \cap N(p)$ containing p. Then we have:

(a) $f \mid C_p$ is one-to-one.

(b) $\varphi(\pi(A_1)) = \{\mathrm{Id}\ R^n\}$ [**5**].

(a) and (b) imply $C_p = \{p\}$. Then $\{p\}$ is open in $A_1(p)$ and therefore $A_1(p) = \{p\}$, which contradicts $\dim A_1(p) \geqslant 1$. □

This result applies in particular to the 3-symmetric spaces classified by A. Gray in [**1**] and to the homogeneous regular s-manifolds (G, H, σ) classified by A. Fedenko [**3**, p. 172].

For example we have

COROLLARY. *If M is a noncompact simply-connected indecomposable Riemannian 3-symmetric space and G is the group of holomorphic isometries (it is transitive on M), then M admits no isometric equivariant imbedding into a finite-dimensional representation of G.* □

For symmetric spaces this result is contained in [**5**].

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