INTEGRAL FORMULAS AND HYPERSPHERES
IN A SIMPLY CONNECTED SPACE FORM

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Abstract. Let \( M^n \) denote a connected compact hypersurface without boundary contained in Euclidean or hyperbolic \( n + 1 \) space or in an open hemisphere of \( S^{n+1} \). We show that if two consecutive mean curvatures of \( M \) are constant then \( M \) is in fact a geodesic sphere. The proof uses the generalized Minkowski integral formulas for a hypersurface of a complete simply connected space form. These Minkowski formulas are derived from an integral formula for submanifolds in which the ambient Riemannian manifold \( \overline{M} \) possesses a generalized position vector field; that is a vector field \( Y \) whose covariant derivative is at each point a multiple of the identity. In addition we prove that if \( M \) is complete and connected with the covariant derivative of \( Y \) exactly the identity at each point then \( M \) is isometric to Euclidean space.

Let \((M, g)\) denote a connected compact oriented \( n \) dimensional Riemannian manifold without boundary with \( h: M \to \overline{M} \) an isometric immersion of \( M \) into an oriented \( n + p \) dimensional Riemannian manifold \((\overline{M}, \overline{g})\). If \( p = 1 \) we let \( \sigma_k \) denote the \( k \)th mean curvature of \( M \). That is \( \sigma_k \) is equal to the \( k \)th elementary symmetric function of the principal curvatures of \( M \) divided by \( \binom{n}{k} \).

Theorem 1. Suppose \( p = 1 \) and assume \( \sigma_k, \sigma_{k+1} \) are constant for some \( k = 1, 2, \ldots, n - 1 \).

(a) If \( \overline{M} = \mathbb{R}^{n+1} \) with the standard metric then \( h \) embeds \( M \) as an \( n \)-sphere in \( \mathbb{R}^{n+1} \).

(b) If \( \overline{M} = \mathbb{H}^{n+1} \) (hyperbolic \( n + 1 \) space) then \( h \) embeds \( M \) as a geodesic \( n \)-sphere in \( \mathbb{H}^{n+1} \).

(c) If \( \overline{M} = S^{n+1} \) with the standard metric and \( h(M) \) is contained in an open hemisphere then \( h \) embeds \( M \) as an \( n \)-sphere in \( S^{n+1} \).

The additional hypothesis of (c) cannot be omitted since there exist nonspherical compact minimal \((\sigma_1 = 0)\) hypersurfaces of \( S^{n+1} \) with constant scalar curvature (which implies \( \sigma_2 \) is constant).

Part (a) of Theorem 1 was proved by Robert Gardner [2] using the generalized Minkowski integral formulas of Hsiung [4]. Gardner’s argument may be applied to prove (b) and (c) once we have the Minkowski formulas for hypersurfaces of \( \mathbb{H}^{n+1} \) and \( S^{n+1} \) respectively. To describe these integral formulas we need the notion of a

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generalized position vector field. Let \( \nabla \) denote the Riemannian connection of \( \overline{M} \) and suppose \( Y \) is a vector field on \( \overline{M} \).

**Definition 1.** \( Y \) is a generalized position vector field of \( \overline{M} \) provided \( \nabla Y = fI \) where \( f \) is a smooth function on \( \overline{M} \) and \( I \) denotes the identity section of \( \text{End}(TM) \).

**Examples.** (i) If \( \overline{M} = \mathbb{R}^{n+p} \), \( \bar{g} \) is the standard flat metric and \( Y = \sum x_i \partial / \partial x_i \) is the ordinary position vector field then \( \nabla Y = I \).

(ii) Let \( \overline{M} = S^{n+p} \) with \( \bar{g} \) the standard metric of constant curvature 1 and let \( X \) denote a parallel vector field on \( \mathbb{R}^{n+p+1} \). Define a vector field \( Y \) on \( S^{n+p} \) by letting \( Y(x) \) equal the orthogonal projection of \( X(x) \) onto \( T_x S^{n+p} \). Then \( \nabla Y = (X \cdot N)I \) where \( N \) denotes the inward unit normal vector field on \( S^{n+p} \).

More generally suppose \( X \) is a parallel vector field on a Riemannian manifold \( (\hat{M}, \langle \, , \rangle , \hat{g}) \) and \( \overline{M} \) is a totally umbilic submanifold of \( \hat{M} \) with \( \bar{g} \) the induced metric. If we define a vector field \( Y \) on \( \overline{M} \) as above then \( \nabla Y = \langle X, \eta \rangle I \) where \( \eta \) denotes the mean curvature normal vector field of \( \overline{M} \). In particular if \( \overline{M} = \hat{M} \) then \( \nabla Y = 0 \).

(iii) Let \( \overline{M} = H^{n+p} =\{ (x_1, x_2, \ldots, x_{n+p}) \in \mathbb{R}^{n+p} \mid x_{n+p} > 0 \} \) with \( \sigma = (1/x_{n+p}^2)\sum dx_i^2 \) and \( Y = \partial / \partial x_{n+p} \). Then \( \nabla Y = -I \).

(iv) Choose any smooth metric on \( S^{n+p-1} \subset \mathbb{R}^{n+p} \). We extend this metric to \( \overline{M} = \mathbb{R}^{n+p} \)-origin by requiring the standard radial vector field \( \partial / \partial r \) of \( \mathbb{R}^{n+p} \)-origin to be of unit length and orthogonal to \( S^{n+p-1} \) and by also requiring parallel translation along radial directions to be ordinary Euclidean parallel translation. Then for \( Y = r \partial / \partial r \) we have \( \nabla Y = I \). Furthermore with this metric \( \overline{M} \) is flat if and only if the original metric gave \( S^{n+p-1} \) constant curvature 1.

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The second ingredient necessary to describe the integral formulas is the notion of a Codazzi tensor. Suppose \( S \) is a tensor of type \((k, k)\) on \( M \) which is alternating in the first \( k \) and in the last \( k \) indices. Then at each point \( x \in M \), \( S_x \) may be identified with an element of \( \text{End} \Lambda^k T_x M \) and we write \( S \in \Gamma[\text{End} \Lambda^k(TM)] \). Given \( S \in \Gamma[\text{End} \Lambda^k(TM)] \) and \( T \in \Gamma[\text{End} \Lambda^j(TM)] \) define \( S \ast T \in \Gamma[\text{End} \Lambda^{k+j}(TM)] \) by wedging together the respective covariant and contravariant components of \( S \) and \( T \). The multiplications \( \ast \) is associative and commutative. Let \( \nabla \) denote the Riemannian connection on the full tensor algebra of \( M \).

**Definition 2.** \( S \in \Gamma[\text{End} \Lambda^k(TM)] \) is a Codazzi tensor of type \((k, k)\) provided

\[
0 = \sum (-1)^{j+1} \left( \nabla_{X_j} S \right) \left( X_1 \wedge X_2 \wedge \cdots \wedge X_{j-1} \wedge X_{j+1} \wedge \cdots \wedge X_{k+1} \right)
\]

for all \( C^\infty \) vector fields \( X_1, X_2, \ldots, X_{k+1} \).

It is routine to show that if \( S \) and \( T \) are Codazzi tensors of types \((k, k)\) and \((j, j)\) respectively then \( S \ast T \) is a Codazzi tensor of type \((k+j, k+j)\).

**Examples.** (a) The identity section \( I_k \) of \( \text{End} \Lambda^k(TM) \) is trivially a Codazzi tensor of type \((k, k)\) since \( \nabla I_k = 0 \).

(b) Suppose \( p = 1 \), \( \overline{M} \) has constant sectional curvature and \( N \) is a unit normal vector field on \( M \). The shape operator \( T \) of \( M \) is a tensor of type \((1, 1)\) on \( M \) defined
by \( T(X) = -\nabla_X N \) where \( X \) denotes a tangent vector to \( M \). Since \( \overline{M} \) has constant sectional curvature the Codazzi equations for \( M \) imply \( T \) is a Codazzi tensor of type \((1,1)\). As a consequence \( S = T^k = T * T * \cdots * T \) \((k \text{ times})\) is a Codazzi tensor of type \((k,k)\) and \( \text{trace } S = k! \left( \overline{e} \right)^{\otimes k} \sigma_k \) (the factor of \( k! \) occurs because of our particular normalization of wedge products).

(c) The curvature operators \( R_{2k} \in \Gamma[\text{End } \Lambda^{2k}(TM)] \) are defined by

\[
\langle R_2(X \wedge Y), Z \wedge W \rangle = \langle R(X, Y)Z, W \rangle
\]

and

\[
R_{2k} = R_2 * R_2 * \cdots * R_2 \quad \text{\((k \text{ times})\)}
\]

where \( R \) denotes the curvature tensor of \( M \) and \( \langle , \rangle \) denotes the extension of \( g \) to the exterior algebra of \( M \). The second Bianchi identity implies \( R_2 \) is a Codazzi tensor of type \((2,2)\) and thus \( R_{2k} \) is a Codazzi tensor of type \((2k,2k)\).

(d) Suppose \( \mathcal{T} \) is a codimension \( k \) distribution on \( M \) and \( \mathcal{K} \) is the distribution orthogonal to \( \mathcal{T} \). Let \( P \) denote orthogonal projection onto \( \mathcal{K} \). Then \( P \) is a Codazzi tensor of type \((1,1)\) if and only if \( \mathcal{T} \) and \( \mathcal{K} \) are parallel distributions (that is invariant under parallel translation along any curve). \( S = P^k \) is a Codazzi tensor of type \((k,k)\) if and only if \( \mathcal{T} \) is integrable with totally geodesic leaves and \( \mathcal{K} \) is “minimal” in the sense that \( P(\Sigma \nabla e_i e_i) = \Sigma \nabla e_i e_i \) whenever \( e_1, e_2, \ldots, e_k \) is a local orthonormal frame field for \( \mathcal{K} \). For example if \( M = S^3 \) and \( \mathcal{T} \) is the tangent space to the fibres of the Hopf fibration \( S^3 \to S^2 \) then \( S = P^2 \) is a Codazzi tensor of type \((2,2)\) even though \( P \) itself is not a Codazzi tensor.

Assume \( Y \) is a generalized position vector field on \( \overline{M} \) and define a tensor \( A \) of type \((1,1)\) on \( M \) by the equation \( g(A(X), Z) = \overline{g}(II(X, Z), Y) \) where \( II \) denotes the second fundamental form of \( M \).

**THEOREM 2.** If \( S \) is a Codazzi tensor of type \((k,k)\) on \( M \) then

\[
\int_M (n - k) f \text{trace } S + \text{trace } S * A \ dV = 0.
\]

**PROOF.** Assume first that \( S \) is a Codazzi tensor of type \((n - 1, n - 1)\). Define an \( n - 1 \) form \( \alpha \) on \( M \) by \( \alpha = Y^{\text{tan}} \ dV \) where \( Y^{\text{tan}} \) denotes the orthogonal projection of \( Y \) onto \( TM \). It is routine to show that if \( X \) is any smooth vector field on \( M \) then \( \nabla_X \alpha = [fX + A(X)] \ dV \). Let \( e_1, e_2, \ldots, e_n \) denote a local oriented orthonormal frame field for \( M \) with \( E_j = e_1 \wedge e_2 \wedge \cdots \wedge e_{j-1} \wedge e_{j+1} \wedge \cdots \wedge e_n \). If we consider \( \omega = \alpha \circ S \) as an \( n - 1 \) form then

\[
d\omega(e_1, e_2, \ldots, e_n) = \sum (-1)^{j+1}(\nabla_{e_j} \omega) E_j
\]

\[
= \sum (-1)^{j+1}(\nabla_{e_j} \alpha) S(E_j) + \alpha \left[ \sum (-1)^{j+1}(\nabla_{e_j} S) E_j \right]
\]

\[
= \sum (-1)^{j+1}[f e_j + A(e_j)] \ dV S(E_j)
\]

\[
= f \text{trace } S + \text{trace } S * A.
\]

Since \( \int_M d\omega = 0 \) the result follows in this case.
Suppose now that \( S \) is a Codazzi tensor of type \((k, k)\) and apply the above result to \( S \ast I_{n-k-1} \). Then

\[
0 = \int_M f \text{trace} S \ast I_{n-k-1} + \text{trace} S \ast A \ast I_{n-k-1} \, dV
\]

which implies

\[
0 = \int_M (n - k) f \text{trace} S + \text{trace} S \ast A \, dV. \quad \square
\]

Theorem 2 was proved in [1] for the case \( M = \mathbb{R}^{n+p} \) with the standard metric and with \( Y \) the ordinary position vector field. By making different choices of \( M, Y \) and \( S \) we get a number of interesting integral formulas. For example if \( S = I \) Theorem 2 becomes

\[
0 = \int_M f + g(\eta, Y) \, dV
\]

where \( \eta \) denotes the mean curvature normal vector field of \( M \). As a consequence \( M \) cannot be minimally immersed into the open subset of \( M \) defined by \( f \neq 0 \). In particular we immediately obtain the (well-known) result that there are no compact minimal submanifolds of Euclidean or hyperbolic space. Similarly there exist no compact minimal submanifolds contained in an open hemisphere of \( S^{n+p} \). As another application if we let \( M = \mathbb{R}^{n+p} \) with the standard metric and let \( Y = \partial / \partial x_i \), \( i = 1, 2, \ldots, n + p \), then Theorem 2 implies

\[
\int_M \eta \, dV = 0.
\]

Finally we derive the generalized Minkowski formulas for a compact hypersurface in a simply connected space form.

**Corollary 3 (Minkowski Formulas).** Assume \( p = 1 \), \( N \) is a unit normal vector field on \( M \) and \( p = \tilde{g}(N, Y) \).

(a) If \( M = \mathbb{R}^{n+1} \) with the standard metric and \( Y = \sum x_i \partial / \partial x_i \) then

\[
\int_M \sigma_k + \rho \sigma_{k+1} \, dV = 0, \quad k = 0, 1, \ldots, n - 1,
\]

where \( \sigma_0 \equiv 1 \).

(b) If \( M = S^{n+1} \) and \( Y \) equals the orthogonal projection of \(-\partial / \partial x_1 \) onto \( S^{n+1} \) then

\[
\int_M x_i \sigma_k + \rho \sigma_{k+1} \, dV = 0, \quad k = 0, 1, \ldots, n - 1.
\]

(c) If \( M = H^{n+1} \) and \( Y = \partial / \partial x_{n+1} \) then

\[
\int_M -x_{n+1} \sigma_k + \rho \sigma_{k+1} \, dV = 0, \quad k = 0, 1, \ldots, n - 1.
\]

**Proof.** For each choice of \( M \) the shape operator \( T \) is a Codazzi tensor of type \((1, 1)\) and \( A = \rho T \). Let \( S = T^k \), \( k = 0, 1, \ldots, n - 1 \) \((T^0 \equiv I)\) and apply Theorem 2. \( \square \)
The formulas of part (a) were proved by Hsiung in [4]. Using the above integral formulas and Gardner's argument [2] we now prove Theorem 1.

**Proof of Theorem 1.** Since $M$ is compact there exists a point on $M$ at which all principal curvatures are positive. Thus $\sigma_k \neq 0, \sigma_{k+1} \neq 0$ and it follows that $\sigma_k^2 - \sigma_{k-1}\sigma_{k+1} \geq 0$ with equality if and only if $M$ is totally umbilic [3]. Our hypotheses imply that for appropriate $f$ and $\rho$

$$\int_M f\sigma_j + \rho\sigma_{j+1} dV = 0, \quad j = 0, 1, \ldots, n - 1,$$

where $f \neq 0$ on $M$. Consequently

$$0 = \sigma_k \int_M f\sigma_k + \rho\sigma_{k+1} dV = \int_M f\sigma_k^2 + \rho\sigma_k\sigma_{k+1} dV$$

and

$$0 = \sigma_{k+1} \int_M f\sigma_{k-1} + \rho\sigma_k dV = \int_M f\sigma_{k-1}\sigma_{k+1} + \rho\sigma_k\sigma_{k+1} dV.$$

Subtracting the second equation from the first yields

$$0 = \int_M f(\sigma_k^2 - \sigma_{k-1}\sigma_{k+1}) dV.$$

Since $f \neq 0$ on $M$, $\sigma_k^2 - \sigma_{k-1}\sigma_{k+1} \equiv 0$ and $M$ is totally umbilic. For each choice of $M$ the only compact totally umbilic hypersurfaces are the geodesic spheres and Theorem 1 follows. $\square$

In light of Theorem 2 it is natural to ask what Riemannian manifolds possess a generalized position vector field such that $f \equiv 1$. Examples (i) and (iv) give two such manifolds, the first of which is complete and the second of which is noncomplete. In fact, we have the following result.

**Theorem 4.** Suppose $(\bar{M}, \bar{g})$ is a complete connected Riemannian manifold and $Y$ is a vector field on $\bar{M}$ such that $\nabla Y = I$. Then $\bar{M}$ is isometric to Euclidean space and under this isometry $Y$ corresponds to the ordinary position vector field of Euclidean space.

**Proof.** The unit vector field $E = Y/\|Y\|$ is defined on the open subset $\{q \in \bar{M} \mid Y(q) \neq 0\}$. Since $\nabla_E E = [(1 - E[\|Y\|/\|Y\|])E$ must be orthogonal to $E$, it follows that $E[\|Y\|] = 1$ and that the integral curves of $E$ are geodesics of $\bar{M}$. Thus if $\gamma_q(t)$ denotes the unique geodesic such that $\gamma_q(\|Y(q)\|) = E(q)$ then $Y(\gamma_q(t)) = t\gamma_q'(t)$ for $t \geq 0$ since $\bar{M}$ is complete. In particular $Y$ has at least one zero on $\bar{M}$. Since $\text{Hess}\|Y\|^2 = 2\bar{g}$ the function $\|Y\|^2$ is strictly convex and therefore $Y$ has exactly one zero $p \in \bar{M}$. By considering a normal neighborhood of $p$ we conclude that if $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = p$ then $Y(\gamma(t)) = t\gamma'(t), t \geq 0$. This implies the exponential mapping $\exp: T_p\bar{M} \to \bar{M}$ is one to one. It is onto by the completeness of $\bar{M}$. Furthermore since

$$\bar{R}(X, Z)Y = -\nabla_X\nabla_Z Y + \nabla_Z \nabla_X Y + \nabla_{[X, Z]} Y = -\nabla_X Z + \nabla_Z X + [X, Z] = 0$$
it follows from the Jacobi equation that the exponential mapping is nonsingular. Thus exp: \( T_pM \to \overline{M} \) is a diffeomorphism and we make the identification 
\((T_p\overline{M}, \exp^* \overline{g}) = (\overline{M}, \overline{g})\).

Let \((r, \theta_1, \theta_2, \ldots, \theta_N)\), \(N + 1 = \dim \overline{M}\), denote a system of spherical coordinates on the inner product space \((T_p\overline{M}, \overline{g}_p)\). Then \(E = \partial/\partial r, \ Y = r\partial/\partial r\) with

\[
\frac{\partial}{\partial \theta_i} = \frac{1}{r^2} \frac{\partial}{\partial \theta_i}, \quad \frac{\partial}{\partial \theta_i} \frac{\partial}{\partial r} = \frac{1}{r^2} \frac{\partial^2}{\partial \theta_i \partial r}.
\]

Therefore the vector fields \(\partial/\partial r, (1/r) \partial/\partial \theta_i, i = 1, 2, \ldots, N\), are parallel in the radial directions with respect to the metric \(\exp^* \overline{g}\). On the other hand these vector fields are clearly parallel in the radial directions with respect to the flat metric \(\overline{g}\) induced on \(T_p\overline{M}\) via \(\overline{g}_p\) and the natural identification \(T_q(T_p\overline{M}) \cong T_p\overline{M}\). Since \(\overline{g} = \exp^* \overline{g}\) at the origin of \(T_p\overline{M}\) it follows that \(\overline{g} = \exp^* \overline{g}\) and \(\overline{M}\) is isometric to Euclidean \(N + 1\) space. \(\square\)

Theorem 4 is equivalent to a result of Tashiro [5] which we now describe. Tashiro shows that if \(\phi\) is a smooth function on a complete Riemannian manifold \((\overline{M}, \overline{g})\) such that \(\text{Hess}\phi = 2c\overline{g}, 0 \neq c \in \mathbb{R}\), then \(\overline{M}\) is isometric to Euclidean space. Since \(\nabla Y = I\) implies \(\text{Hess}\|Y\|^2 = 2\overline{g}\), Theorem 4 follows as a corollary. Conversely, given such a function \(\phi\) then the vector field \(Y = 1/2c \text{ grad } \phi\) satisfies \(\nabla Y = I\) and Tashiro's result may be derived from Theorem 4.

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