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COMPLETE HYPERSURFACES WITH $RS = 0$ IN E^{n+1}

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ABSTRACT. A locally symmetric Riemannian manifold satisfies $RR = 0$ and in particular $RS = 0$. The purpose of this paper is to show that the conditions $RR = 0$ and $RS = 0$ are equivalent for complete hypersurfaces in E^{n+1} and to give by $RS = 0$ some characterizations of locally symmetric hypersurfaces in E^{n+1}.

If a Riemannian manifold M is locally symmetric, then its curvature tensor R satisfies

$$R(X, Y) \cdot R = 0 \tag{0.1}$$

for any tangent vectors X and Y, where the endomorphism $R(X, Y)$ operates on R as a derivation of the tensor algebra at each point of M. Nomizu **[5]** proved the following:

Let M be a connected and complete Riemannian n-manifold which is isometrically immersed in a Euclidean space E^{n+1} so that the type number $k(x) \geqslant 3$ at least at one point x. If M satisfies condition (0.1), *then it is of the form $M = S^k \times E^{n-k}$, where S^k is a hypersurface in a Euclidean subspace E^{k+1} of E^{n+1} and E^{n-k} is a Euclidean subspace orthogonal to E^{k+1}.*

Let S be the Ricci tensor of M. Then the condition (0.1) implies in particular

$$R(X, Y) \cdot S = 0 \tag{0.2}$$

for any tangent vectors X and Y. Then Tanno **[8]** showed the following results:

(1) *For hypersurfaces in E^{n+1} with the positive scalar curvature, the conditions* (0.1) *and* (0.2) *are equivalent. By using* (1),

(2) *Let M be a connected and complete Riemannian n-manifold which is isometrically immersed in a Euclidean space E^{n+1} so that the type number $k(x) \geqslant 3$ at least at one point x. If M satisfies the condition* (0.2) *and has the positive scalar curvature, then it is of the form $M = S^k \times E^{n-k}$.*

These were generalized by Ryan **[7]** in the case of hypersurfaces with the nonnegative scalar curvature or constant scalar curvature.

The purpose of this paper is to prove the following

THEOREM. *For complete hypersurfaces in E^{n+1}, the conditions* (0.1) *and* (0.2) *are equivalent.*

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1. Lemmas. We shall assume that M is oriented and that the second fundamental form A has three distinct eigenvalues $\lambda(x)$, $\mu(x)$ and 0 which have constant multiplicities p ($\geqslant 2$), q ($\geqslant 2$) and $n-p-q$ (>0), respectively. We define three distributions on M as follows:

$$T_\lambda(x) = \{X \in T_x(M) \mid AX = \lambda(x)X\},$$
$$T_\mu(x) = \{X \in T_x(M) \mid AX = \mu(x)X\},$$
$$T_0(x) = \{X \in T_x(M) \mid AX = 0\}.$$

We have $T_x(M) = T_\lambda(x) + T_\mu(x) + T_0(x)$ (direct sum). For any $Z \in T_x(M)$, Z_λ, Z_μ, Z_0 will denote the components of Z in $T_\lambda(x)$, $T_\mu(x)$ and $T_0(x)$, respectively. Then we can easily show the following [**2, 5**]

LEMMA 1. *T_λ, T_μ and T_0 are differentiable and involutive.*

From $p \geqslant 2$ (resp. $q \geqslant 2$) we have [**5**]

LEMMA 2. *If X belongs to T_λ (resp. T_μ), then $X \cdot \lambda = 0$ (resp. $X \cdot \mu = 0$).*

Now, let $X \in T_\lambda$, $Y \in T_0$ and computing both sides of the Codazzi equation, we get [**2**]

LEMMA 3. (i) *If $X \in T_\lambda$ and $Y \in T_0$, then $(\nabla_X Y)_\lambda = -((Y \cdot \lambda)/\lambda)X$, where ∇ denotes the covariant differentiation for the Riemannian connection on M.*

(ii) *If $Y \in T_0$, then $\nabla_Y(T_0) \subset T_0$.*

Similarly, we have [**4**]

LEMMA 4. *If $X \in T_\lambda$ and $Y \in T_\mu$, then*

$$(\nabla_X Y)_\lambda = -((Y \cdot \lambda)/(\lambda - \mu))X \quad \text{and} \quad (\nabla_Y X)_\mu = ((X \cdot \mu)/(\lambda - \mu))Y.$$

The following lemma is basic.

LEMMA 5 (CARTAN [**1**]). *Let M be a hypersurface in a space $\tilde{M}^{n+1}(c)$ of constant curvature c, $c \leqslant 0$, whose principal curvatures are constant. Then at most two of them are distinct.*

By Lemma 1, around each point x of M we can choose an orthonormal frame $\{X_1, \ldots, X_p, Y_{p+1}, \ldots, Y_{p+q}, Z_{p+q+1}, \ldots, Z_n\}$ such that $\{X_1, \ldots, X_p\}$, $\{Y_{p+1}, \ldots, Y_{p+q}\}$ and $\{Z_{p+q+1}, \ldots, Z_n\}$ are bases for the distributions T_λ, T_μ and T_0 respectively. Hereafter, we shall use indices a, b, c for the range $1, \ldots, p$; i, j, k for $p+1, \ldots, p+q$ and r, s, t for $p+q+1, \ldots, n$. From the Codazzi equation we have

$$g((\nabla_{X_a} A)Y_i, Z_r) = g((\nabla_{Y_i} A)Z_r, X_a) = g((\nabla_{Z_r} A)X_a, Y_i),$$

i.e.,

$$\mu g(\nabla_{X_a} Y_i, Z_r) = -\lambda g(\nabla_{Y_i} Z_r, X_a) = (\lambda - \mu) g(\nabla_{Z_r} X_a, Y_i), \tag{1.1}$$

for all a, i, r and, hence,

$$\begin{aligned} g(\nabla_{Y_i} Z_r, X_a) g(\nabla_{X_a} Y_i, Z_r) + g(\nabla_{X_a} Y_i, Z_r) g(\nabla_{Z_r} X_a, Y_i) \\ + g(\nabla_{Z_r} X_a, Y_i) g(\nabla_{Y_i} Z_r, X_a) = 0, \end{aligned} \tag{1.2}$$

unless

$$g(\nabla_{Y_i} Z_r, X_a) g(\nabla_{X_a} Y_i, Z_r) g(\nabla_{Z_r} X_a, Y_i) = 0.$$

In terms of Lemma 3(ii) we get

$$g(\nabla_{Z_r} Z_r, X_a) = g(\nabla_{Z_r} Z_r, Y_i) = 0.$$

Now, we assume that $(p-1)\lambda + (q-1)\mu = 0$. Differentiating $(p-1)\lambda + (q-1)\mu = 0$ in each direction, we get

$$g(\nabla_{X_a} X_a, Y_i) = g(\nabla_{Y_i} Y_i, X_a) = 0, \tag{1.3}$$

and

$$g(\nabla_{X_a} X_a, Z_r) = g(\nabla_{Y_i} Y_i, Z_r) \tag{1.4}$$

for all a, i, r. Then (1.2), (1.3), Lemmas 3 and 4 give

$$\begin{aligned} 0 &= g(R(X_a, Z_r) Z_r, X_a) \\ &= Z_r g(\nabla_{X_a} X_a, Z_r) - \sum_s g(\nabla_{Z_r} Z_r, Z_s) g(\nabla_{X_a} X_a, Z_s) \\ &\quad - g(\nabla_{X_a} X_a, Z_r)^2 - 2 \sum_i g(\nabla_{X_a} Y_i, Z_r) g(\nabla_{Z_r} X_a, Y_i), \end{aligned} \tag{1.5}$$

$$\begin{aligned} \lambda\mu &= g(R(X_a, Y_i) Y_i, X_a) \\ &= -\sum_r g(\nabla_{X_a} X_a, Z_r) g(\nabla_{Y_i} Y_i, Z_r) - 2 \sum_r g(\nabla_{X_a} Y_i, Z_r) g(\nabla_{Y_i} Z_r, X_a), \end{aligned} \tag{1.6}$$

$$0 = g(R(X_a, Y_i) Y_j, X_a) = -2 \sum_r g(\nabla_{X_a} Y_j, Z_r) g(\nabla_{Y_i} Z_r, X_a), \tag{1.7}$$

for $i \neq j$,

$$0 = g(R(X_a, Y_i) Y_i, X_b) = -2 \sum_r g(\nabla_{Y_i} X_a, Z_r) g(\nabla_{X_b} Z_r, Y_i), \tag{1.8}$$

for $a \neq b$. Hence by a similar argument to Proposition 2.1 of [**3**] we obtain

Lemma 6. *If $(p-1)\lambda + (q-1)\mu = 0$ and M is complete, then $g(\nabla_{X_a} Y_i, Z_r) \neq 0$ for some a, i, r.*

Proof. Suppose $g(\nabla_{X_a} Y_i, Z_r) \equiv 0$ for all a, i, r. Since a leaf of $\mathcal{K}$ of T_0 is totally geodesic and complete [**2, 3**], choosing Z_r as a unit tangent vector field along a geodesic $L(s)$ of $\mathcal{K}$, we can write (1.5) as

$$Z_r g(\nabla_{X_a} X_a, Z_r) = g(\nabla_{X_a} X_a, Z_r)^2. \tag{1.5$'$}$$

Note that we may assume $\lambda > 0$ and that

$$g(\nabla_{X_a} X_a, Z_r) = Z_r(\log \lambda)$$

is considered as a function on $L(s)$. Hence $g(\nabla_{X_a} X_a, Z_r) \equiv 0$ or $g(\nabla_{X_a} X_a, Z_r) \equiv (s_0 - s)^{-1}$ for some constant s_0. Combining Lemmas 2–5, (1.3) and (1.4), the former cannot occur. If the latter holds, then $g(\nabla_{X_a} X_a, Z_r)$ is not defined at $s = s_0$, which contradicts completeness.

From (1.7) and (1.8), we get $g((\nabla_{X_a} Y_i)_0, (\nabla_{X_a} Y_j)_0) = 0$ and $g((\nabla_{X_a} Y_i)_0, (\nabla_{X_b} Y_i)_0) = 0$ for $i \neq j$ and $a \neq b$, using (1.1). On the other hand, $|(\nabla_{X_a} Y_i)_0| = |(\nabla_{X_b} Y_j)_0|$ for all a, i, b, j follows from (1.6). Let $\mathcal{C}_a$ be the q-dimensional subspace of T_0 spanned by $(\nabla_{X_a}Y_i)_0$, $i = p+1,\ldots,p+q$, and $\mathfrak{D}_i$ be the p-dimensional subspace of T_0 spanned by $(\nabla_{X_a}Y_i)_0$, $a = 1,\ldots,p$, on an open subset $G = \{x \in M; \Sigma_r g(\nabla_{X_a}Y_i, Z_r)^2 \neq 0 \text{ at } x\}$. Then we have [3]

Lemma 7. *Under the assumption of* $(p-1)\lambda + (q-1)\mu = 0$, $\mathcal{C}_a = \mathfrak{D}_i$ *for all* a, i.

2. Proof of Theorem. Our conditions (0.1) and (0.2) reduce respectively to

$$\lambda_i\lambda_j\lambda_k(\lambda_i - \lambda_j) = 0 \tag{2.1}$$

and

$$\lambda_i\lambda_j(\lambda_i - \lambda_j)(\operatorname{trace} A - \lambda_i - \lambda_j) = 0 \tag{2.2}$$

[6, 7]. Assume (2.2). If the type number $k(x) \leqslant 2$ for any $x \in M$, then (2.1) is automatically satisfied. Hence we may suppose the type number $\geqslant 3$ at some point, say, $0 \in M$. Let λ and μ be distinct nonzero principal curvatures at 0. If ν is a principal curvature distinct from λ and μ, we have

$$\nu(\operatorname{trace} A - \lambda - \nu) = 0, \quad \nu(\operatorname{trace} A - \mu - \nu) = 0.$$

Since $\lambda \neq \mu$ we must conclude that $\nu = 0$. But if this is true, then trace $A = \lambda + \mu$. On the other hand, trace $A = p\lambda + q\mu$, where p and q are the appropriate multiplicities. Thus, $(p-1)\lambda + (q-1)\mu = 0$ and p and q are greater than 1, since $k(0) \geqslant 3$.

If $p + q = n > 2$, the standard argument of [6] shows that λ and μ are constant near 0. Thus, $\lambda\mu = 0$, which implies a contradiction. Thus, at most two principal curvatures are distinct and (2.1) holds. Hence we may assume $p + q < n$.

Let $W = \{x \mid k(x) \geqslant 3\}$, which is an open set. Let W_0 be the connected component of 0 in W. By the above argument we see that either

(2.3) A has only one eigenvalue λ,

(2.4) A has two distinct principal curvatures λ and 0, or

(2.5) A has three distinct principal curvatures λ, μ and 0

holds at 0 and then on W_0. If we assume (2.3) on W_0, then W_0 is umbilic. Hence λ is constant on W_0. Next, assume that (2.5) holds on W_0. Then we know that $k(x)$, p and q are constant on W_0 and $\lambda(x)$ and $\mu(x)$ are differentiable functions. Then, since $(p-1)\lambda + (q-1)\mu = 0$ holds, Lemmas 1–7 are valid. Moreover, by a similar argument to Proposition 2.1 of [3] we know that (2.5) cannot occur.

In fact, by Lemmas 6 and 7, we know $p = q \ (=: p_0) \leqslant n - p - q$. Let $\mathcal{C} = \mathcal{C}_a$. Since we have

$$0 = g(R(X_a, X_b)Y_j, X_a) = \sum_r g(\nabla_{X_b}Y_j, Z_r)g(\nabla_{X_a}Z_r, X_a),$$

$(\nabla_{X_a}X_a)_0$ is orthogonal to $\mathcal{C}$, or

$$g(\nabla_{X_a}X_a, Z_\rho) = 0, \quad \text{for } 2p_0 + 1 \leqslant \rho \leqslant 3p_0,$$

using a basis Z_ρ, $\rho = 2p_0 + 1,\ldots,3p_0$ of $\mathcal{C}$. Suppose $n - p - q > p_0$ and $\mathcal{E}$ is the orthogonal complement of $\mathcal{C}$ in T_0 on G. Let Z_σ, $3p_0 + 1 \leqslant \sigma \leqslant n$, be a basis of $\mathcal{E}$. Then from

$$0 = g(R(X_a, Z_\sigma)Z_\tau, Y_i) = \sum_\rho g(\nabla_{Z_\sigma}Z_\tau, Z_\rho)g(\nabla_{X_a}Z_\rho, Y_i),$$

for $\sigma, \tau \geqslant 3p_0 + 1$, we obtain

$$\tilde{\nabla}_{Z_\sigma} Z_\tau = \sum_{\omega = 3p_0+1}^{n} g(\nabla_{Z_\sigma} Z_\tau, Z_\omega) Z_\omega,$$

where $\tilde{\nabla}$ denotes the covariant differentiation for the Riemannian connection on E^{n+1}. Thus $\mathfrak{S}$ is an involutive distribution on G whose leaf is totally geodesic. For $Z_\sigma \in \mathfrak{S}$, let $L(s)$ be the geodesic whose tangent vector is Z_σ. Note that $L(s)$ can be extended completely even if $g(\nabla_{Y_i} Z_r, X_a) = 0$ at some point on it, since a leaf of T_0 is complete. Moreover

$$Z_\sigma g(\nabla_{X_a} X_a, Z_\sigma) = g(\nabla_{X_a} X_a, Z_\sigma)^2 \tag{1.5)''$$

holds for all $s \in E^1$ by (1.5). Then contradiction is shown in the same way as the proof of Lemma 6. Therefore we conclude $n - p - q = p_0$ and $g(\nabla_{X_a} X_a, Z_r) = 0$ for all r. Thus, by means of Lemma 5, (2.5) cannot occur.

Hence either (2.3) or (2.4) holds on W_0. If (2.4) holds on W_0, then the same argument as [**5**] shows λ is a constant function on W_0. Now assume (2.3) (resp. (2.4)) holds on W_0. We show that W_0 is actually equal to M. Suppose $W_0 \neq M$ and let x be a point of $\overline{W_0} - W_0$. By the continuity argument for the characteristic polynomial of A, we see that (2.3) (resp. (2.4)) holds at x. Thus W_0 is open and closed so that $W_0 = M$ and thus (2.1) is satisfied on M.

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