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AN EFFECTIVE VERSION OF HALL'S THEOREM

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ABSTRACT. Manaster and Rosenstein [**1972**] constructed a recursively bipartite highly recursive graph that satisfies Hall's condition for a bipartite graph to have a matching, but has no recursive matching. We discuss a natural extension of Hall's condition which assures that every such graph has a recursive matching.

A *bipartite graph* $G = (A, B, E)$ consists of a set of vertices partitioned into A and B and a set E of edges such that $E \subset A \times B$. A *matching* from A to B is a function $m\colon A \to B$ such that for all $a \in A$, $(a, m(a)) \in E$. For any $X \subset A$ let $N_G(X) = \{b \in B\colon$ for some $a \in A$, $(a, b) \in E\}$ and $s_G(X) = |N_G(X)| - |X|$. When G is clear from the context we shall drop the subscript.

Phillip Hall proved the following

THEOREM 0 (P. HALL [**1935**]). *If $G = (A, B, E)$ is a finite bipartite graph then there is a matching from A into B if for all $X \subset A$, $s(X) \geqslant 0$.*

Marshall Hall provided the following extension of this theorem to infinite graphs. Let $X \subset\subset A$ denote that X is a finite subset of A.

THEOREM 1 (M. HALL [**1948**]). *If G is a bipartite graph such that for every $a \in A$, $N(\{a\})$ is finite then G has a matching from A into B iff for all $X \subset\subset A$, $s(X) \geqslant 0$.*

We shall say that G satisfies the Hall condition, denoted H.c., if for all $X \subset\subset A$, $s(X) \geqslant 0$. Theorem 1 is somewhat unsatisfying because, while it asserts the existence of a matching, it does not provide an algorithm for constructing that matching. In fact, in a sense made precise by the following theorem there may be no such algorithm.

THEOREM 2 (MANASTER AND ROSENSTEIN [**1972**]). *There is a recursively bipartite, highly recursive graph $G = (A, B, E)$ which satisfies the H.c. but does not have a recursive matching.*

Some explanation is in order. Roughly speaking, a graph G is *recursive* if there are algorithms for computing both its vertex set and its edge set; G is *highly recursive* if it is recursive, every vertex has finite degree and there is an algorithm for computing these degrees; G is *recursively bipartite* if it is bipartite, say $G = (A, B, E)$, and there is an algorithm for determining membership in A and B. Thus a locally finite

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bipartite graph G is a recursively bipartite, highly recursive graph if there are algorithms for determining all "local" information about G. A matching m is *recursive* if there is an algorithm for computing it. For a more detailed and very readable account of recursive functions, the reader is referred to §1 of Manaster and Rosenstein [**1972**].

In this paper we provide a condition, similar in spirit to the H.c., which guarantees that a recursively bipartite, highly recursive graph $G = (A, B, E)$ will have a recursive matching from A to B. We also show that this condition cannot be relaxed by constructing a recursively bipartite, highly recursive graph that satisfies "most" of the condition but does not have a recursive matching. This construction is similar to the construction in Manaster and Rosenstein [**1973**] but requires a different argument to show that it accomplishes its objective. Finally, we use our condition to derive a noneffective result for countable graphs with infinite degree. The reader is referred to Holz, Podewski, and Steffens [**1979**] for further results of this nature.

DEFINITION. Let $G = (A, B, E)$ be a bipartite graph. G satisfies the *expanding Hall condition*, denoted e.H.c. iff there exists a function $h: N \to N$ such that $h(0) = 0$ and for all $X \subset\subset A$, $h(n) \leqslant |X|$ implies $n \leqslant s(X)$. G satisfies the *recursive expanding Hall condition*, denoted r.e.H.c., iff G satisfies the e.H.c. and there is a recursive function h that witnesses this.

Notice that since $h(0) = 0$, the e.H.c. implies the H.c.

THEOREM 3. *If $G = (A, B, E)$ is a recursively bipartite highly recursive graph which satisfies the r.e.H.c., then G has a recursive matching m of A into B.*

PROOF. Let h witness the r.e.H.c. for G. Let a_0 be the first element of A. Form an induced subgraph $G_0 = (A_0, B_0, E_0)$ of G by taking $A_0 \cup B_0$ to be the set of vertices a distance of at most $2h(1) + 1$ from a_0. This can be done effectively and G_0 is finite because G is highly recursive. Using the finite version of Hall's Theorem, construct a matching m_0 of A_0 into B_0. Let $m(a_0) = m_0(a_0) = b_0$. It suffices to show that $G' = G - \{a_0, b_0\}$ is still a recursively bipartite, highly recursive graph which satisfies the r.e.H.c., for then we can just iterate the above process. The first part is immediate, so we consider the r.e.H.c. We begin by showing that G' satisfies the H.c. Let $X \subset\subset A' = A - \{a_0\}$. If $X \subset A_0$ then $s_{G'}(X) \geqslant 0$ since m_0 is a matching of A_0 into B_0. If $b_0 \notin N(X)$ then $s_{G'}(X) = s_G(X) \geqslant 0$. Finally suppose that $(b_0, a_1) \in E$, $a_1 \in X$ and there exists $a_2 \in X - A_0$. Without loss of generality X is connected. Thus by the choice of G_0 there is a path of length at least $2h(1) + 1$ from a_1 to a_2; so $|X| \geqslant h(1) + 1 > h(1)$ and $s_{G'}(X) \geqslant s_G(X) - 1 \geqslant 0$. Now let $h'(n) = h(n + 1)$ for all $n > 0$. Clearly, if $|X| \geqslant h'(n)$ then $|X| \geqslant h(n + 1)$, $s_G(X) \geqslant n + 1$, and $s_{G'}(X) \geqslant n$.

The author thanks Alfred Manaster for his helpful suggestions for presenting the proof of Theorem 3. The r.e.H.c was first formulated by the author to prove the following result.

COROLLARY 4. *Let $G = (A, B, E)$ be a highly recursive, bipartite graph such that every vertex in A has degree greater than n and every vertex in B has degree less than or equal to n. Then G has a recursive matching.*

PROOF. First note that G is recursively bipartite since we can decide membership in A or B via the recursive degree function. So by the theorem it suffices to find a recursive function h that witnesses the r.e.H.c. for G. Using the fact that if $X \subset\subset A$ then $(n+1)|X| \leqslant n|N(X)|$, it is easy to check that $h(x) = nx$ is such a function.

For the proof of the next theorem we shall need to make use of the usual effective listing $\phi_0, \phi_1, \ldots$ of all algorithmically computable functions. We write $\phi_e^k(x) = y$ if the eth algorithm stops after k steps when started on x and gives output y; $\phi_e^k(x)$ is not defined if the eth algorithm does not stop in k steps when started on y. Again the reader is referred to §1 of Manaster and Rosenstein [**1972**] for more details.

THEOREM 5. *There exists a highly recursive, recursively bipartite graph $G = (A, B, E)$ which satisfies the e.H.c. but does not have a recursive matching of A into B.*

PROOF. Fix a recursive partition $\{P_e : e \in N\}$ of N into infinitely many infinite recursive pieces. Our graph G will be the disjoint union of infinitely many independent graphs $G_e = (A_e, B_e, E_e)$, $e \in N$, which satisfy the following conditions. We denote $A_e \cup B_e$ by V_e.

(0) $A_e, B_e \subset P_e$;

(1) ϕ_e is not a matching of A_e into B_e;

(2) for all $X \subset\subset A_e$, if $|V_e| = \infty$ then $|X| \leqslant s(X)$;

(3) for all $X \subset\subset A_e$, $\min(e, |X|) \leqslant s(X)$.

Suppose that we have accomplished this. Since any matching of A into B is a matching of $A \cap X$ into $B \cap X$ for any component X, by (1) G does not have a recursive matching from A to B. To see that G satisfies the e.H.c., we fix an arbitrary m and determine n such that for all $X \subset\subset A$, $n \leqslant |X|$ implies that $m \leqslant s(X)$. In light of Theorem 1, we certainly cannot do this effectively. Let $D = \{e : e < m$ and $|V_e| < \infty\}$, $d = |\bigcup_{i \in D} V_i|$, and $n = m + d$. If $X \subset\subset A$ and $n \leqslant |X|$, then,

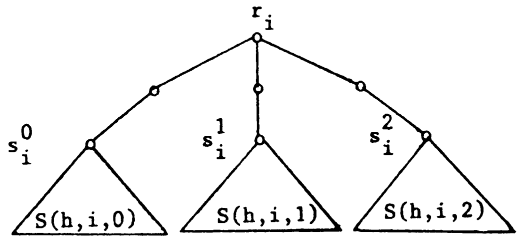
$$\begin{aligned} s(X) &= \sum_{i \in N} s(X \cap V_i) \\ &\geqslant \sum_{\substack{i<m \\ i \notin D}} s(X \cap V_i) + \sum_{i \geqslant m} s(X \cap V_i) \\ &\geqslant \sum_{\substack{i<m \\ i \notin D}} |X \cap V_i| + \sum_{i \geqslant m} \min\bigl(m, |X \cap V_i|\bigr) \\ &\geqslant \min\bigl(m, |X| - d\bigr) \geqslant m. \end{aligned}$$

In order to describe the construction of G_e we must introduce some notation. Let $F(h, e) = (A_F, B_F, E_F)$ be the forest consisting of $3e$ trees $T_i(h)$, $i < 3e$, of the form in the diagram where each $S(h, i, j)$ is a full binary tree of height $2h + 1$ whose root is s_i^j. Let $L(k, i, j)$ be the vertices of $S(h, i, j)$ at level k. (The root s_i^j is at level 0.) Let

$$A_f = \bigcup_{\substack{k \leqslant h \\ i < 3e \\ j < 3}} L(2k, i, j) \cup \{r_i : i < e\}$$

and

$$B_f = \bigcup_{\substack{k \leq h \\ i<3e \\ j<3}} L(2k+1, i, j).$$



We shall construct G_e in stages. $G_e = (A_e^h, B_e^h, E_e^h)$ will be the part of G_e constructed by the end of the hth stage. If the nth element of P_e is a vertex in V_e, it will be in $V_e^n = A_e^n \cup B_e^n$, G_e^h will be an induced subgraph of G_e, and no vertex introduced at stage $h+1$ will be adjacent to any vertex introduced before stage h. Assuming that the construction of the G_e^h's will be effective, the first two conditions assure that G_e is recursive and recursively bipartite and this together with the last condition assures that G_e is highly recursive. Let G_e^0 be isomorphic to $F(0, e)$ where V_e^0 consists of the first $3e \cdot 13$ elements of P_e and the assignment of vertices of V_e^0 to vertices of $F(0, e)$ is made in some reasonable fashion. Now proceeding inductively, suppose that we have defined G_e^h.

Case 1. If ϕ_e^h is not a matching of $\{r_0, \ldots, r_{3e-1}\}$ into B_e, let C_e^{h+1} be an extension of G_e^h to an isomorphic copy of $F(h+1, e)$ formed in some reasonable way from the next $9e(2^{2h} + 2^{2h+1})$ elements of P_e.

Case 2. Suppose h is the first natural number such that ϕ_e^h is a matching from $\{r_0, \ldots, r_{3e-1}\}$ into B_e. For each $i < 3e - 1$ let $j(i)$ be such that $(\phi_e^h(r_i), s_s^{j(i)}) \in E_e$. Let I be the next $e2^{2(h+1)}$ elements in P_e. Now form G_e^{h+1} by making each vertex of I adjacent to each vertex of $\bigcup_{i<e} L(2h+1, i, j(i))$ and putting the vertices of I into A_e.

Case 3. Otherwise let $G_e^{h+1} = G_e^h$.

Now we check (1)–(3). For (1), notice that if we never enter Case 2 then ϕ_e is not a matching of A_e into B_e. So suppose at stage $h+1$ we are in Case 2. Consider the subgraph H of G_e induced by $I = \bigcup_{i<3e} S(h, i, j(i))$. By the choice of $j(i)$, if ϕ_e is a matching of A_e into B_e, then ϕ_e is a matching of $H \cap A_e$ into $H \cap B_e$. But this is impossible since

$$|H \cap A_e| = 3e \sum_{i=0}^{h} 2^{2i} + e2^{2(h+1)} = e \cdot 2 \cdot 4^{h+1} - e$$

and

$$|H \cap B_e| = 3e \sum_{i=0}^{h} 2^{2i+1} = e \cdot 2 \cdot 4^{h+1} - 2e.$$

To prove (2), we remark that if $|V_e| = \infty$ then G_e is a forest and every vertex in A_e has degree 3. Thus for $X \subset\subset A_e$, $s(X) \geq |X|$. Finally suppose $X \subset\subset A_e$ where

$|V_e| < \infty$. Since $|V_e| < \infty$ we reached Case 2 in the construction of G_e at some stage h. Notice that if $X \cap I = \varnothing$ then just as for (2), $S(X) \geqslant |X|$. So suppose $X \cap I \neq \varnothing$. Let H' be the graph induced by the vertices of H together with $\{\phi_e(r_i): i < 3e\}$. Let F be $G_e - H'$. Then F is a forest with roots r_i, for $i < 3e$, and every vertex in $F \cap X$ has degree 3 or is a root of degree 2. Thus $s_F(F \cap X)$ is greater than or equal to $|X|$. So it suffices to show that $s(H' \cap X)$ is at least $\min(e, |X|)$. Each $\phi_e(r_i)$ is adjacent only to $s_i^{j(i)}$ in H'. Thus $s(H' \cap X) \geqslant s(U \cap X)$ where U is the set of vertices of H with the roots $s_i^{j(i)}$ removed. Since each element of $L(2h+1, i, j(i))$ is adjacent to each element of I, if $|U \cap X| < e$ then

$$s'(U \cap X) \geqslant e \cdot 2^{2(h+1)} - e \geqslant 3e \geqslant |U \cap X|.$$

Now suppose $|U \cap X| \geqslant e$. Thus it suffices to show that $e \leqslant s(U \cap X)$. We can count $N(U \cap X)$ by counting $L(2h+1, i, j(i))$ together with the elements of $L(2k-1, i, j(i))$ that are adjacent to elements of $L(2k, i, j(i)) \cap X$ for $0 < k \leqslant h$. Since elements of $L(2k-1, i, j(i))$ are adjacent to only two elements of $L(2k, i, j(i))$ and every element of $L(2k, i, j(i))$ is adjacent to some element of $L(2k-1, i, j(i))$,

$$N(U \cap X) \geqslant 3e^{2h+1} + \tfrac{1}{2}|(U - I) \cap X|$$

and

$$\begin{aligned} s(U \cap X) &\geqslant 3e2^{2h+1} - \tfrac{1}{2}|(U - I) \cap X| - |I \cap X| \\ &\geqslant 6e4^h - \frac{1}{2} \cdot 3e \sum_{i=1}^{h} 4^i - 4e4^h \geqslant 2e. \end{aligned}$$

THEOREM 6. *Let $G = (A, B, E)$ be a countable bipartite graph. If G satisfies the e.H.c. then G has a matching from A into B.*

PROOF. Let a_0 be the first element of A. As in the proof of Theorem 3, it suffices to match a_0 to $b_0 \in B$ in such a manner that $G' = G - \{a_0, b_0\}$ satisfies the e.H.c. As before the only difficulty is in checking that G' satisfies the H.c. We must consider two cases: (1) $\delta(a) < \infty$ and (2) $\delta(a) = \infty$. For (1), let $A_f = \{a \in A: \delta(a) < \infty\}$ and G_f be the graph induced by $A_f \cup N(A_f)$. By Theorem 1, G_f has a matching m from A to B. Clearly, letting $b_0 = m(a_0)$ endows G' with the H.c. For case (2), let A_0 be the maximum subset of A such that $s(A_0) = 0$. The e.H.c. assures that A_0 exists and is finite. Let b_0 be the least element of $N(\{a_0\}) - N(A_0)$. Again it is easy to check that G' satisfies the H.c.

REFERENCES

M. Hall (1948) *Distinct representatives of subsets*, Bull. Amer. Math. Soc. **54**, 922–926.

P. Hall (1935) *On representatives of subsets*, J. London Math. Soc. **10**, 26–30.

M. Holz, K.-P. Podewski and K. Steffens (1979) *Hall families and the marriage problem*, J. Combinatorial Theory Ser. A **27**, 161–180.

A. Manaster and J. Rosenstein (1972) *Effective matchmaking (recursion theoretic aspects of a theorem of Philip Hall)*, Proc. London Math. Soc. **25**, 615–654.

______ (1973) *Effective matchmaking and k-chromatic graphs*, Proc. Amer. Math. Soc. **39**, 371–378.

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