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A CORRECTION NOTE ON "GENERALIZED HEWITT-SAVAGE THEOREMS FOR STRICTLY STATIONARY PROCESSES"

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ABSTRACT. Conditions on the distribution of a process $\{X_n, n \in I\}$ are given under which the invariant, tail and exchangeable σ-fields coincide; the index set I is either the positive integers or all the integers. The results proven here correct similar statements given in [3].

1. Let $\{X_n, n \in I\}$ be a sequence of real-valued r.v.'s on the probability space $(\mathfrak{R}^\infty, \mathfrak{B}^\infty, P)$, let $\mathfrak{I}$, $\mathfrak{T}$, and $\mathfrak{E}$ be the invariant, tail, and exchangeable σ-fields (see [3] for definitions and terminology), and consider the case where I is the set of positive integers J.

It is well known (see [2, p. 39; or 4]) that without reference to the probability P, the following strict inclusions always hold:

$$\mathfrak{I} \subset \mathfrak{T} \subset \mathfrak{E}. \tag{1}$$

Hence, for any probability P:

$$\mathfrak{I} \subset \mathfrak{T} \subset \mathfrak{E}(P). \tag{2}$$

Looking at (1) and (2) one can see that Theorem 1 in [3] is erroneous. The inaccuracies in [3] stem from not considering separately the case where I is J, the positive integers, and the case where I is Z, the integers.

2. Z setup. In this case one can define $\mathfrak{I}$ and $\mathfrak{E}$ as before mutatis mutandis (now T is onto as well as 1-1, and the permutations move around a finite number of possibly negative and positive coordinates); there are, however, several σ-fields that could merit being called "tail σ-field". (For a discussion of these σ-fields, and many more things related to this note and to [1], see [4].) We will be satisfied here considering $\mathfrak{T}$ to be $\bigcap_{n=1}^{\infty} \sigma(X_i, |i| \geq n)$, where $\sigma(X_i, i \in I)$ denotes the σ-field generated by the variables X_i, $i \in I$.

In this setup it is known that

$$\mathfrak{T} \subset \mathfrak{E}. \tag{3}$$

The inclusion is strict and no other inclusion is valid among $\mathfrak{I}$, $\mathfrak{T}$, $\mathfrak{E}$ in this setup (see [4]). From (3) it is obvious that for any probability P:

$$\mathfrak{T} \subset \mathfrak{E}(P). \tag{4}$$

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3. Now we will give conditions under which the inclusions (2) and (4) can be reversed.

Let $T_n \in \Sigma$ be defined in the J setup by: $(T_n\omega)_k = (\omega)_k$ for $k \geqslant n+1$; $(T_n\omega)_k = (\omega)_{k+1}$, $1 \leqslant k \leqslant n-1$; $(T_n\omega) = (\omega)_1$. And in the Z setup by: $(T_n\omega)_k = (\omega)_k$ for $|k| \geqslant n+1$; $(T_n\omega)_k = (\omega)_{k+1}$, $|k| \leqslant n-1$; $(T_n\omega)_n = (\omega)_{-n}$; $(T_n\omega)_{-n} = (\omega)_n$.

It is easily seen that $T_n^{-1}C = T^{-1}C$ for every cylinder $C \in \sigma(X_1, \ldots, X_{n-1})$ in the J setup and for every cylinder $C \in \sigma(X_i, |i| \leqslant n-1)$ in the Z setup.

Let $P \circ T^{-n}$, P_n be the measures on $\mathfrak{B}^\infty$ defined by $(P \circ T^{-n})A = P(T^{-n}A)$ and $P_n(A) = P(T_n^{-1}A)$ for $n = 1, 2, \ldots$.

Let $\ll$ denote absolutely continuity of measures.

Theorem 1. *In the J setup, if $P \circ T^{-1} \ll P$ and $P_n \ll P$ uniformly in n, then*

$$\mathfrak{I} = \mathfrak{T} = \mathfrak{S}(P).$$

Proof. It is enough to prove $\mathfrak{S} \subset \mathfrak{I}(P)$. Let $A \in \mathfrak{S}$ and let C be a cylinder in $\sigma(X_1, \ldots, X_{n-1})$ for n to be determined later. We have

$$P(A\Delta T^{-1}A) = P\big(A\Delta T_n^{-1}C\big) + P\big(T_n^{-1}C\Delta T^{-1}A\big) = P\big(T_n^{-1}A\Delta T_n^{-1}C\big) + P(T^{-1}C\Delta T^{-1}A)$$
$$= P\big(T_n^{-1}(A\Delta C)\big) + P\big(T^{-1}(A\Delta C)\big).$$

Let $\varepsilon > 0$ be arbitrary. Find δ (independent of n) such that $P(G) < \delta$ implies $P(T_n^{-1}G) < \varepsilon/2$ and $P(T^{-1}G) < \varepsilon/2$.

Then $P(A\Delta T^{-1}A) < \varepsilon$. Hence $P(A\Delta T^{-1}A) = 0$, i.e., $A = T^{-1}A(P)$.

Theorem 2. *In the Z setup, if $P \circ T^{-n} \ll P$ and $P_n \ll P$ both uniformly in n, then $\mathfrak{I} = \mathfrak{T} = \mathfrak{S}(P)$.*

Proof. It suffices to prove (i) $\mathfrak{S} \subset \mathfrak{I}(P)$ and (ii) $\mathfrak{I} \subset \mathfrak{T}(P)$. The proof of (i) is the same as in Theorem 1 mutatis mutandis. For (ii), let $A \in \mathfrak{I}$ and $\varepsilon > 0$ be arbitrary. Find δ such that $P(T^{-n}B) < \varepsilon$ for all n whenever $P(B) < \delta$ and a cylinder $C \in \sigma(X_i, |i| \leqslant m)$ such that $P(A\Delta C) < \delta$. Then $P(A\Delta T^{-m}C) + P(T^{-m}(A\Delta C)) < \varepsilon$ and hence $P(A\Delta T^{-m}C) = 0$. Consider $D = T^{-m}C$. $T^{-n}D \in \sigma(X_i, |i| \geqslant n)$. Take $E = \limsup T^{-n}D$. Then $E \in \mathfrak{T}$ and $P(A\Delta E) = 0$. This finishes the proof.

4. In proving Theorems 1 and 2 we have not used the assumption in [**3**]:

(5) $\quad$ for each $\sigma \in \Sigma$, $P(\sigma^{-1}A) = 0$ when $P(A) = 0$.

An example is given there, where supposedly

(6) $\quad$ $\mathfrak{S} \subset \mathfrak{I} \subset \mathfrak{T}(P)$ but $\mathfrak{S} = \mathfrak{I} = \mathfrak{T}(P)$ does not hold

because (5) is not fulfilled.

The example is the following: consider the probability measure P determined by assigning probability $1/2$ to each of the sequences $(1, 0, 1, 0, \ldots)$ and $(0, 1, 0, 1, \ldots)$. To see that (6) is incorrect, think of P as a two-state homogeneous Markov chain with (stationary) initial distribution $\pi(0) = \pi(1) = 1/2$, and transition probabilities $p_{00} = p_{11} = 0$, $p_{01} = p_{10} = 1$. Clearly this chain has one ergodic class $\{0, 1\}$ and two periodic classes $\{0\}$ and $\{1\}$ of states.

In [**1**], Blackwell and Freedman (see also Freedman [**2**]) characterize $\mathfrak{I}$, $\mathfrak{T}$ and $\mathfrak{S}$ when X_n is a homogeneous recurrent countable Markov chain. Applying those results in our case (regardless of the value of $\pi(0)$ and $\pi(1)$ insofar as $0 < \pi(0) < 1$) it is plain to see that $\mathfrak{I} = \text{trivial}(P)$, whereas $\mathfrak{T} = \mathfrak{E}(P) =$ the σ-field generated by the two one-point atoms $\{(1,0,1,0,\ldots)\}$ and $\{(0,1,0,1,\ldots)\}$, so $\mathfrak{I} \underset{\neq}{\subset} \mathfrak{T} = \mathfrak{E}(P)$ and (6) is invalid.

Note that this Markov chain, though strictly stationary, does not satisfy the hypothesis $P_n \ll P$ required in Theorem 1 of [3] because the set $\{\omega\} = \{(1,0,1,0,\ldots,1,0,\underline{0},1,0,1,0,\ldots)\}$ (where _ denotes the nth position) has P-measure 0, but since $T_n^{-1}\omega = (0,1,0,1,0,\ldots)$, $\{\omega\}$ has $P \circ T_n^{-1}$-measure $1/2$.

5. Using this characterization of $\mathfrak{I}, \mathfrak{T}, \mathfrak{E}$ for the Markov chain case, we can detect an error in the proof of Theorem 2 in [3], where it is claimed that if f is the indicator of an $\mathfrak{E}$-set, then Tf is also in $\mathfrak{E}$, i.e., if A is exchangeable, $T^{-1}A$ is exchangeable. To see that this is not the case, even modulo P, where P is a probability under which X_n is strictly stationary, consider the example of [2, p. 46]: a Markov chain $\{X_n, n \geqslant 1\}$ with three states, whose nonzero transition probabilities are $p_{12} = p_{23} = 1$, $p_{31} = p_{32} = 1/2$. $\mathfrak{E}$ is nontrivial, in fact its P-atoms are $\{X_1 = 3\}$ and $\{X_1 \in \{1,2\}\}$, and $T^{-1}\{X_1 = 3\} = \{X_1 = 2\}(P)$, and this latter set does not belong to $\mathfrak{E}$.

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