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NONINCREASE ALMOST EVERYWHERE OF CERTAIN MEASURABLE FUNCTIONS WITH APPLICATIONS TO STOCHASTIC PROCESSES[1]

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ABSTRACT. Let $x(t)$, $0 \leqslant t \leqslant 1$, be a real valued measurable function having a local time $\alpha_{[0,t]}(x)$, $0 \leqslant t \leqslant 1$. If the latter is continuous in t for almost all x, then almost every t is not a point of increase of the function $x(\cdot)$.

1. Introduction. Let $x(t)$, $0 \leqslant t \leqslant 1$, be a real valued measurable function. For every pair of linear Borel sets A and I, where $I \subset [0,1]$, define $\nu(A, I) =$ Lebesgue measure of $\{t: t \in I, x(t) \in A\}$. If, for fixed I, $\nu(\cdot, I)$ is absolutely continuous as a measure on sets A, then its Radon-Nikodým derivative, denoted $\alpha_I(x)$, is called the local time of $x(t)$ relative to I. The theme of our previous research in this area is the relation between the smoothness of the function $\alpha_{[0,t]}(x)$ in its variables and the irregularity of the original function $x(t)$. See, for example, [**1, 2**]. While such results are valid for arbitrary measurable functions, they are of particular interest in the study of random functions arising in the theory of stochastic processes because the relevant properties of the local time happen to be directly deducible from the underlying probabilistic data, namely, the finite-dimensional distributions of the process.

It has been shown in a series of papers by Geman and Horowitz that several of my earlier results were actually valid under conditions of greater generality than originally given; furthermore, weaker conclusions follow under conditions of much greater generality. A full survey of this area is given in [**5**].

Here we discuss another improvement of the latter type. Geman introduced a certain hypothesis on the local time which is easily applicable to a large class of functions arising as sample functions of stochastic processes [**4**]. It is the assumption that for almost all x, $\alpha_{[0,t]}(x)$ is continuous as a function of $t, 0 \leqslant t \leqslant 1$. (In [**5**] it is called "AC-0".) He showed that, under this hypothesis,

(a) $$\operatorname{approx.lim}_{s \to t} \left| \frac{x(s) - x(t)}{s - t} \right| = \infty, \quad \text{for almost all } t.$$

(b) The set $\{s: x(s) = x(t)\}$ is uncountable for almost all t.

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These correspond to earlier results of the author where the hypothesis was the joint continuity of the local time [**1**, **2**].

The purpose of this paper is to show that Geman's hypothesis implies another qualitative result about the erratic nature of the original function $x(t)$: Almost every t is not a point of increase of $x(\cdot)$. The validity of such a result was suggested by the classical result of Dvoretzky, Erdös and Kakutani [**3**] on the nonincrease everywhere of the Brownian motion process. Many years ago I found a simple, real variable proof of their theorem based on local time, using the following observations. If $x(\cdot)$ has a point of increase at t, then for some a and b, with $a < t < b$, $x(s) \leqslant x(t)$ for $a \leqslant s \leqslant t$, and $x(s) \geqslant x(t)$ for $t \leqslant s \leqslant b$. If $\alpha_{[0,s]}(x)$ is continuous in x for each s, and right continuous in s for each x, then it follows from the definition of the local time that $\alpha_{[a,t]}(x(t)) = \alpha_{[t,b]}(x(t)) = 0$, and so, by additivity, $\alpha_{[a,b]}(x(t)) = 0$. Thus the set of points of increase of $x(\cdot)$ is included in the set

$$\bigcup_{I:\ I \text{ has rational endpoints}} \{t\colon t \in I, \alpha_I(x(t)) = 0\}.$$

Assume now that $x(\cdot)$ is continuous and nowhere constant, and that its local time $\alpha_I(\cdot)$ is positive on the interior of the range of $x(s)$, $s \in I$, for every interval I with rational endpoints. Then any point of increase in the set displayed above is necessarily a minimum or maximum relative to some I, and so $x(\cdot)$ would have to be constant on some I, which is a contradiction. Hence, such a function has no points of increase. Now the Brownian sample function has all the properties required above: It is continuous and nowhere constant, and has a jointly continuous local time [**7**] which is positive on the interior of the range [**6**].

The proof of our current result is based on the ideas above. Instead of joint continuity of α, we require only continuity in t for almost all x. Instead of the strong positivity property of the Brownian path, we use the general fact, valid for all functions with local times, that $\alpha_I(x(t)) > 0$ for almost all $t \in I$. The conclusion is, naturally, weaker, namely, that the set of points of increase is of measure 0. The result also holds for points of decrease; indeed, the points of decrease for $x(t)$, $0 \leqslant t \leqslant 1$, are the points of increase for $x(1-t)$, $0 \leqslant t \leqslant 1$.

While the results above are valid for arbitrary measurable functions, they are of special interest for the sample functions of stochastic processes. Indeed, Geman [**4**] has given simple sufficient conditions on the bivariate distributions of the process for the continuity of α in t.

2. The main result. Let $x(t)$, $0 \leqslant t \leqslant 1$, be a real valued measurable function and $\alpha_I(x)$ its local time. Let $\mathcal{K}$ be the class of all subintervals of [0, 1] with rational endpoints. According to a basic property of local time [**1**], there exists a version of it such that:

(i) *For each* x, $\alpha_I(x)$ *is a measure on the Borel sets* I *of* [0, 1].

(ii) *For each interval* $J \in \mathcal{K}$, $\alpha_J(x) = 0$ *if* x *does not belong to the closure of the range of* $x(t)$, $t \in J$.

Such a version is called *regular*.

The local time is called temporally continuous if the function $\alpha_{[0,t]}(x)$ is continuous in t for almost all x.

LEMMA 1. *If there exists a version of the local time which is temporally continuous, then every version is; in particular, a regular version is temporally continuous.*

PROOF. Let α be any version, and let α' be a temporally continuous version. As two versions of the same Radon-Nikodým derivative, $\alpha_J(x)$ and $\alpha'_J(x)$ agree except on an x-set of measure 0, for each $J \in \mathcal{K}$; therefore, there exists a set N whose complement is of measure 0 such that $\alpha_{[0,t]}(x) = \alpha'_{[0,t]}(x)$ for all x in N, for all rational t. Since α and α' are monotonic in t for all x in N, and they coincide for rational t, it follows that the continuity of α' implies the continuity of α.

We recall that a function $f(x)$ is approximately continuous at x_0 if

$$(2\varepsilon)^{-1}\,\mathrm{mes}(x: |x - x_0| < \varepsilon, |f(x) - f(x_0)| > \delta) \to 0, \qquad \varepsilon \to 0,$$

for every $\delta > 0$. As a consequence of the countability of $\mathcal{K}$ and a fundamental property of measurable functions, we have

(iii) *There exists a set A_1 of measure 0 such that $\alpha_J(x)$ is approximately continuous on ${}^cA_1$ for every $J \in \mathcal{K}$.*

DEFINITION 1. *A point t in the interior of an interval I is said to split the image of I if for all $s \in I$, we have $x(s) \leqslant x(t)$ or $x(s) \geqslant x(t)$ accordingly as $s \leqslant t$ or $s \geqslant t$.*

DEFINITION 2. *A point t in* $[0, 1]$ *is a point of increase for $x(\cdot)$ if it splits the image of some interval $I \subset [0, 1]$.*

It is obvious that

(iv) *t is a point of increase if and only if it splits the image of some $J \in \mathcal{K}$.*

LEMMA 2. *Let $\alpha_I(x)$ be temporally continuous. For every $J \in \mathcal{K}$, $\alpha_J(x(t)) = 0$ for almost all $t \in J$ which split the image of J.*

PROOF. By the definition of temporal continuity, there is a set A_2 of measure 0 such that $\alpha_{[0,t]}(x)$ is continuous in t for all $x \in {}^cA_2$. Let A_1 be as in (iii), and define $A = A_1 \cup A_2$. For arbitrary rational $a < b$, put $J = [a, b]$, and suppose that t splits the image of J and also that $x(t) \in {}^cA$. For every $\delta > 0$, there exist rational numbers c and d with $a < c < t < d < b$ and $d - c < \delta$ such that $\alpha_{[a,c]}(x(t)) = \alpha_{[d,b]}(x(t)) = 0$. To prove this, we note that the closure of the range of $x(s)$, $a \leqslant s \leqslant c$, is contained in $(-\infty, x(t)]$; hence, by Proposition (ii) above, $\alpha_{[a,c]}(x) = 0$ for every $x > x(t)$. Since $x(t) \in {}^cA$, it follows by approximate continuity that $\alpha_{[a,c]}(x(t)) = 0$. Indeed, for any $\delta > 0$, if $\alpha_{[a,c]}(x(t)) > \delta$, then

$$\begin{aligned}1 &= \varepsilon^{-1}\,\mathrm{mes}(x: x(t) < x < x(t) + \varepsilon)\\ &= \varepsilon^{-1}\,\mathrm{mes}\big(x: x(t) < x < x(t) + \varepsilon, |\alpha_{[a,c]}(x) - \alpha_{[a,c]}(x(t))| > \delta\big)\\ &\leqslant 2(2\varepsilon)^{-1}\,\mathrm{mes}\big(x: |x(t) - x| < \varepsilon, |\alpha_{[a,c]}(x) - \alpha_{[a,c]}(x(t))| > \delta\big)\end{aligned}$$

and the latter tends to 0 for $\varepsilon \to 0$, which is a contradiction. The same argument holds for $\alpha_{[d,b]}(x(t))$.

The membership of $x(t)$ in ${}^cA_2$ now implies that $\alpha_{[a,t]}(x(t)) = \alpha_{[t,b]}(x(t)) = 0$. Since α_I is additive in I (Proposition (i) above), it follows that $\alpha_J(x(t)) = \alpha_{[a,t]}(x(t)) + \alpha_{[t,b]}(x(t)) = 0$.

Put $S = \{t: t \in J, t$ splits the image of $J\}$. It has just been shown that $S \cap x^{-1}({}^cA) \subset \{t: t \in J, \alpha_J(x(t)) = 0\}$. The complementary set, $S \cap x^{-1}(A)$, has measure 0; indeed, since A has measure 0, $x^{-1}(A)$ also has measure 0 because $x(\cdot)$ has a local time.

THEOREM 1. *If the local time is temporally continuous, then almost every point t is not a point of increase for $x(\cdot)$.*

PROOF. The set $\{t: t \in J, \alpha_J(x(t)) = 0\}$ is known to have measure 0 [**5**, formula (6.7)]. Hence, by Lemma 2, the set of points in J which split the image of J has measure 0. Proposition (iv) now implies that the set of points of increase is of measure 0.

REFERENCES

1. S. M. Berman, *Gaussian processes with stationary increments: local times and sample function properties*, Ann. Math. Statist. **41** (1970), 1260–1272.

2. ______, *Gaussian sample functions: uniform dimension and Hölder conditions nowhere*, Nagoya Math. J. **46** (1972), 63–86.

3. A. Dvoretzky, P. Erdös and S. Kakutani, *Nonincrease everywhere of the Brownian motion process*, Proc. Fourth Berkeley Sympos. Math. Statist. and Prob., vol. 2, Univ. of California Press, Berkeley, 1961, pp. 103–116.

4. D. Geman, *A note on the continuity of local times*, Proc. Amer. Math. Soc. **57** (1976), 321–326.

5. D Geman and J. Horowitz, *Occupation densities*, Ann. Probab. **8** (1980), 1–60.

6. D. B. Ray, *Sojourn times of diffusion processes*, Illinois J. Math. **7** (1963), 615–630.

7. H. F. Trotter, *A property of Brownian motion paths*, Illinois J. Math. **2** (1958), 425–432.

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