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# UNIFORM CONVERGENCE OF DISTRIBUTION FUNCTIONS

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ABSTRACT. Necessary and sufficient conditions are given for uniform convergence of probability distribution functions.

Weak convergence is the standard mode of convergence used for probability distribution functions. This is due partly to the Levy continuity theorem, which connects the weak convergence of distributions to the pointwise convergence of their characteristic functions. Nevertheless, there are advantages in knowing distributions converge uniformly.

In this paper we clarify the relation between weak and uniform convergence and show that uniform convergence can also be characterized in terms of a mode of convergence of characteristic functions.

Let $F_n$ and $F$ denote right continuous probability distribution functions, $\mu_n$ and $\mu$ the corresponding measures, and $\phi_n$ and $\phi$ the corresponding characteristic functions. Weak convergence of $F_n$ to $F$ is denoted $F_n \overset{w}{\to} F$ and indicates that $F_n(t) \to F(t)$ at continuity points $t$ of $F$. It is well known (see [**1**, p. 260]) that if $F_n \overset{w}{\to} F$ and $\mu_n(\{x\}) \to \mu(\{x\})$ for each $x$, then $F_n(x) \to F(x)$ uniformly in $x$.

Levy's continuity theorem states that weak convergence of probability distribution functions $F_n$ to $F$ is equivalent to pointwise convergence of their characteristic functions $\phi_n$ to $\phi$. In order to characterize uniform convergence of probability distribution functions in terms of the convergence of characteristic functions we introduce a norm on those functions $f$ such that $\lim_{T\to\infty}(1/2T)\int_{-T}^{T}|f(t)|^2\,dt$ exists. This limit is denoted $\|f\|^2$.

LEMMA. *$\|\int e^{itx}\,d\nu(x)\|^2$ exists for finite real measures $\nu$ and equals $\sum|\nu(\{x\})|^2$.*

This is a simple extension of Wiener's formula.

THEOREM. *The following are equivalent*:
(1) $F_n \to F$ *uniformly*,
(2) $F_n \overset{w}{\to} F$ *and* $\sum|\mu_n(\{x\}) - \mu(\{x\})|^2 \to 0$,
(3) $\phi_n(t) \to \phi(t)$ *for all* $t$ *and* $\|\phi_n - \phi\| \to 0$.

PROOF. (1) $\Leftrightarrow$ (2) is straightforward.
(2) $\Leftrightarrow$ (3) by the Levy continuity theorem and the lemma. □
Dyson's theorem [**2**; **3**, p. 349] is a simple corollary.

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COROLLARY. *If $\phi_n(t) \to \phi(t)$ uniformly in $t$ then $F_n(x) \to F(x)$ uniformly in $x$.*

Kawata [**3**, p. 352] states that the converse of Dyson's theorem is also true. But this is false.

Let $\mu_n$ be the uniform distribution on $\{0, 1/n, 2/n, \ldots, (n-1)/n\}$ and $\mu$ be uniform on [0, 1]. Then $F_n \to F$ uniformly, but $\phi_n(t)$ is periodic so $\limsup \phi_n(t) = 1$ while $\lim \phi(t) = 0$. Thus $\sup_t |\phi_n(t) - \phi(t)| \geqslant 1$ for all $n$.

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