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THE LEVY-LINDEBERG CENTRAL LIMIT THEOREM IN L_p, $0 < p < 1$

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ABSTRACT. A $L_p(T, \Sigma, \mu)$-valued r.v. X, $0 < p < 1$, satisfies the Lévy-Lindeberg central limit theorem if and only if it is centered and pregaussian, that is, if and only if $EX(t) = 0$ μ-a.e. and $\int_T (EX^2(t))^{p/2}\, d\mu(t) < \infty$.

1. Introduction and preliminaries. Let (E, Σ) be a measurable linear space, i.e. a linear space with jointly Σ-measurable sum and product by scalars, and let X be an E-valued random variable. Then we say that X satisfies the Lévy-Lindeberg central limit theorem (CLT) if the probability laws of $\Sigma_{i=1}^n X_i/n^{1/2}$, where X_i, $i \in \mathbf{N}$, are independent copies of X, converge weakly to a Gaussian measure on (E, Σ). A L_p-valued random variable X, $1 \leqslant p \leqslant 2$, satisfies the Lévy-Lindeberg CLT if and only if $EX(t) = 0$ a.s. and $\int_T (EX^2(t))^{p/2}\, d\mu(t) < \infty$, this last condition meaning that X is pregaussian, i.e., that there exists a centered Gaussian probability measure on L_p which has the covariance of X. This fact may be viewed as a consequence of L_p being a cotype 2 Banach space—see e.g. [**1**, §3.8]. It is proved in [**5**] that a symmetric l_p-valued r.v. X, for $0 < p < 1$, satisfies the CLT if and only if $\Sigma_\alpha (EX_\alpha^2)^{p/2} < \infty$, the X_α being the coordinates of X. So, it is natural to ask whether the same happens in L_p, $0 < p < 1$. The difference between l_p and L_p is not trivial; in fact, the lack of nonzero continuous linear functionals on L_p gives some technical interest to the problem. As it turns out, the techniques for the proof of the theorem mentioned in the abstract do not differ much from those used in Banach spaces, and they are based on approximation of X by simple variables which satisfy the CLT; the simple variables we take here are similar to those used in [**4**].

It is also of some interest to realize that the same CLT result holds for both the very pathological L_p spaces, $0 < p < 1$, and the much nicer spaces L_p, $1 \leqslant p \leqslant 2$.

The notation will be as in [**1**]; in particular the probability law of an r.v. X defined on a probability space $(\Omega, \mathfrak{F}, P)$, $P \circ X^{-1}$, will be denoted by $\mathcal{L}(X)$, and $\to_w$ and w-lim will both denote weak convergence of probability measures on metric spaces.

Let (T, Σ, μ) be an arbitrary measure space. Let $\{Z_i\}_{i=1}^\infty$ be $L_p(T, \Sigma, \mu)$-valued random variables (r.v.'s) whose laws are *tight*. The union of the supports of these laws is separable, and therefore, by [**7**, III.8.5], which also holds for $0 < p < 1$, it is contained in $L_p(T_1, \Sigma_1, \mu_1)$, where $T_1 \in \Sigma$, Σ_1 is a sub-σ-algebra of $\Sigma|_{T_1}$ and $\mu_1 = |_{\Sigma_1}$, with the properties that μ_1 is σ-finite and Σ_1 is countably generated (i.e.

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generated by a countable algebra). So, since we will be dealing only with countable sequences of random variables, without loss of generality we may assume that Σ *is countably generated* and that μ *is* σ*-finite*, in particular that the complete metric space $L_p(T, \Sigma, \mu)$ is *separable*.

It is convenient to recall some facts on L_p-valued r.v.'s and their relation to processes defined on T which have almost all of their trajectories in L_p. For this we follow [3]. (1) Every jointly measurable stochastic process $X(t, \omega)$ defined on T and with almost all its trajectories in $L_p(T, \Sigma, \mu)$, induces a $L_p(T, \Sigma, \mu)$-valued r.v. and conversely, for every Borel probability measure ρ on L_p there exists a jointly measurable stochastic process $X(t, \omega)$ with almost all its trajectories in L_p and whose law as a L_p-valued r.v. is ρ. (2) A L_p-valued r.v. X (or p.m. ρ on L_p) is *Gaussian* if for any pair of independent copies of X, X_1 and X_2, the random variables $X_1 + X_2$ and $X_1 - X_2$ are independent; this definition is equivalent to: the process X with trajectories in $L_p(T, \Sigma, \mu)$ is Gaussian if and only if there exists $T_0 \in \Sigma$ with $\mu(T_0) = 0$ such that for all finite sets $\{t_1, \ldots, t_k\} \subset T - T_0$, the random vector $(X(t_1), \ldots, X(t_k))$ is Gaussian. If X is symmetric Gaussian (the only Gaussian L_p-valued r.v.'s that we will consider) then $sX_1 + tX_2$ and $tX_1 - sX_2$, for $s^2 + t^2 = 1$, $s, t > 0$, are independent and distributed like X. Two more properties of Gaussian random variables in L_p: if X is Gaussian then $E\|X\|_p^r < \infty$ for all $r > 0$, where here and elsewhere in this note we denote by $\|x\|_p$ the quasinorm

$$\|x\|_p = \int_T |x(t)|^p \, d\mu(t), \qquad x \in L_p, \quad 0 < p < 1$$

(in fact X has much stronger integrability); and finally: a centered L_p-valued r.v. X is pregaussian if and only if $\int_T (EX^2(t))^{p/2} \, d\mu(t) < \infty$. To these facts we may add the following:

LEMMA 1. *Let Z_n, $n \in \mathbf{R}$, be $L_p(T, \Sigma, \mu)$-valued symmetric Gaussian r.v.'s such that $\mathcal{L}(Z_n) \to_w \mathcal{L}(Z)$. Then Z is also Gaussian and symmetric.*

PROOF. By a now classical result of Skorokhod, there exist, on some probability space, sequences $\{Z_n'\}$, $\{Z_n''\}$ independent of each other such that $\mathcal{L}(Z_n') = \mathcal{L}(Z_n'') = \mathcal{L}(Z_n)$ and $Z_n \to Z'$, $Z_n'' \to Z''$ a.s., where Z' and Z'' are independent and have the law of Z. Then, since $Z_n' + Z_n''$ and $Z_n' - Z_n''$ are independent, so are $Z' + Z''$ and $Z' - Z''$. Since this is a property of the joint distribution of Z' and Z'', it follows that Z is Gaussian. $\square$

Let us also recall the approximation lemma that we will repeatedly use.

LEMMA 2. *Let M be a complete separable metric space and let $Y_{n,\varepsilon}$, Z_n be M-valued r.v.'s ($\varepsilon > 0$, $n \in \mathbf{N}$) such that:* (a) *$\mathcal{L}(Y_{n,\varepsilon}) \to_w \mathcal{L}(Y_\varepsilon)$ for some r.v.'s Y_ε, and* (b) *for each $\varepsilon > 0$ there exists $n(\varepsilon) < \infty$ such that for all $n > n(\varepsilon)$, $d_{\mathrm{pr}}(Z_n, Y_{n,\varepsilon}) < \varepsilon$, where d_{pr} is any distance metrizing convergence in probability. Then* w-$\lim_n \mathcal{L}(Z_n)$ *and* w-$\lim_{\varepsilon \downarrow 0} \mathcal{L}(Y_\varepsilon)$ *exist and are equal* (w-lim *denotes limit in the weak*-topology*).

This lemma is proved by means of an $\varepsilon - \delta$ argument using the fact that the set $\mathcal{P}(M)$ of probability measures on M is a complete metric space for the w^*-topology (the topology of weak convergence of p.m.'s). A weaker version of it appears in [**2**], and was first used in limit theorems in Banach spaces by G. Pisier [**6**].

We will also need the following two lemmas from [**5**].

LEMMA 3 [**5**]. *Let X be a $L_p(T, \Sigma, \mu)$-valued r.v., $0 < p < 1$, such that if X_i, $i \in \mathbf{N}$, are independent copies of X, then*

$$\mathfrak{L}\left(\sum_{i=1}^{n} X_i/n^{1/2}\right) \to_w \gamma$$

for some Borel p.m. γ on L_p. Then, for all $0 < r < 2$,

$$E\left\|\sum_{i=1}^{n} X_i/n^{1/2}\right\|_p^{r/p} \to \int \|x\|_p^{r/p}\, d\gamma(x).$$

PROOF (SKETCH). As pointed out in [**5**, Remark after Theorem 6.9], if $\tilde{X}_i$ are independent symmetrizations of X_i (that is, $\tilde{X}_i = X_i - X_i'$ where $\{X_i, X_i'\}_{i=1}^{\infty}$ are independent copies of X), then, for all $n \in \mathbf{N}$ and $t > 0$,

$$P\left\{\max_{k \leqslant n}\left\|\sum_{i=1}^{k} \tilde{X}_i\right\|_p > t\right\} < 2P\left\{\left\|\sum_{i=1}^{n} \tilde{X}_i\right\|_p > 2^{p-1}t\right\}$$

(which can be obtained by adapting Kahane's proof of Lévy's inequality—see e.g. [**1**, proof of Theorem 3.2.6]). This allows us to use an argument of Pisier [**6**, Proposition 2.1] and prove the lemma as in Exercise 2.6.9 of [**1**], desymmetrization included. □

In fact Lemma 3 generalizes to any domain of attraction—see loc. cit. and the reference there to de Acosta and Giné's article.

LEMMA 4 [**5**]. *Let $\{\xi_i\}_{i=1}^{\infty}$ be independent copies of ξ, a real valued symmetric r.v., let $p > 0$, and define $\delta_n = \inf[u > 0 : nP\{|\xi| > u\} \leqslant 1/8 \cdot 3^p]$. Then there exists $c > 0$ such that for all $n \in \mathbf{N}$,*

$$n^{p/2}\left(E\xi^2 I_{[|\xi| \leqslant \delta_n]}\right)^{p/2} \leqslant CE\left|\sum_{i=1}^{n} \xi_i\right|^p. \tag{2}$$

(Actually, the inequalities in 3.3 and 3.4 [**5**] are much more complete, but we only need this here.)

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2. The CLT. We start proving the CLT in a particular case.

LEMMA 5. *Let Z be a $L_p(T, \Sigma, \mu)$-valued r.v. such that $EZ = 0$ ($EZ(t) = 0$ μ-a.e.) and such that for some $c < \infty$ and $T' \in \Sigma$ with $\mu(T') < \infty$, $|Z(t, \omega)| \leqslant c$ and $Z(t, \omega) = Z(t, \omega) I_{T'}(t)$ $P \times \mu$-a.e. Then, if Z_i, $i \in \mathbf{N}$, are independent copies of Z, we have that*

$$\mathfrak{L}\left(\sum_{i=1}^{n} Z_i/n^{1/2}\right) \to_w \gamma \tag{3}$$

where γ is a symmetric Gaussian p.m. on $L_p(T, \Sigma, \mu)$ such that for all s, t outside a set of μ-measure zero,

$$\int_{L_p} x(s)x(t)\,d\gamma(x) = EZ(s)Z(t). \tag{4}$$

PROOF. Let $\mathcal{Q}$ be a countable algebra of sets generating $\Sigma' = \Sigma\,|_{T'}$ and let $\{\pi_k\}_{k=1}^{\infty}$ be refining collections of finite numbers of disjoint sets $A \in \mathcal{Q}$ such that $\bigcup_{k=1}^{\infty}\pi_k = \mathcal{Q}$. Let $\mu' = \mu\,|_{\Sigma'}$ and $q \geq 1$. Define, for $x \in L_q(T', \Sigma, \mu')$,

$$U_k(x) = \sum_{A \in \pi_k}\left[(\mu(A))^{-1}\int_A x(t)\,d\mu(t)\right] I_A. \tag{5}$$

Then $U_k(x) \to x$ in $L_q(T')$ (hence in L_p for all $p \leq q$) because the same is obviously true for the $\mathcal{Q}$-measurable simple functions, which are dense in $L_q(T', \Sigma', \mu')$. Hence, by bounded convergence,

$$E\int_{T'}(Z - U_k Z)^2\,d\mu \to 0 \quad \text{as } k \to \infty. \tag{6}$$

So, if Z_i are independent copies of Z, we have

$$\begin{aligned}
(7)\quad E\left\|n^{-1/2}\sum_{i=1}^{n}(Z_i - U_k Z_i)\right\|_p^{2/p} &= E\left[\int_{T'}\left|n^{-1/2}\sum_{i=1}^{n}(Z_i - U_k Z_i)\right|^p d\mu\right]^{2/p} \\
&\leq [\mu(T')]^{2/p-1}\int_{T'} E\left[n^{-1/2}\sum_{i=1}^{n}(Z_i - U_k Z_i)\right]^2 d\mu \\
&= [\mu(T')]^{2/p-1}\int_{T'} E(Z - U_k Z)^2\,d\mu \\
&\to 0 \quad \text{uniformly in } n \text{ as } k \to \infty \text{ by (6)}.
\end{aligned}$$

Since the random variables $U_k Z$ take values in a finite-dimensional space (different for each k), it follows by the CLT in $\mathbf{R}^m$, $m = \operatorname{card}(\pi_k)$, that

$$\mathcal{L}\left(\sum_{i=1}^{n} U_k Z_i/n^{1/2}\right) \to_w \gamma_k \tag{8}$$

with γ_k Gaussian symmetric such that

$$\int_{L_p} x(s)x(t)\,d\gamma_k(x) = EU_k Z(s)U_k Z(t) \tag{9}$$

for every $k \in \mathbf{N}$. (In fact, if $(\xi_1, \ldots, \xi_m)$ is a Gaussian symmetric r.v. in $\mathbf{R}^m$ with the same covariance as the random vector $((\mu(A_1))^{-1}\int_{A_1} Z\,d\mu, \ldots, (\mu(A_m))^{-1}\int_{A_m} Z\,d\mu)$, where $\pi_k = \{A_1, \ldots, A_m\}$, then $\gamma_k = \mathcal{L}(\sum_{i=1}^{m}\xi_i I_{A_i})$.) So, Lemmas 1 and 2 together with (7) and (8) prove that

$$\mathcal{L}\left(\sum_{i=1}^{n} Z_i/n^{1/2}\right) \to_w \gamma \tag{3}$$

where γ is the Gaussian measure defined by

$$\gamma = w\text{-}\lim_k \gamma_k. \tag{10}$$

To prove (4) we notice that by Lemma 3 and the limits (3) and (8),

$$E\left\| \sum_{i=1}^{n} U_k Z_i / n^{1/2} \right\|_p^{r/p} \to \int \|x\|_p^{r/p} \, d\gamma_k(x),$$

$$E\left\| \sum_{i=1}^{n} Z_i / n^{1/2} \right\|_p^{r/p} \to \int \|x\|_p^{r/p} \, d\gamma(x), \qquad 0 < r < 2.$$

This, together with (7) implies that

$$\int_{L_p} \|x\|_p^{r/p} \, d\gamma_k(x) \to \int_{L_p} \|x\|_p^{r/p} \, d\gamma(x), \qquad 0 < r < 2. \tag{11}$$

By (10) and (11), there exist Gaussian r.v.'s G_k, G, with laws γ_k, γ, $k \in \mathbf{N}$, such that $G_k \to G$ a.s. and $E\|G_k - G\|_p \to 0$. Then, there exists a subsequence $\{k_l\}$ such that

$$G_{k_l}(t, \omega) \to G(t, \omega) \qquad P \times \mu\text{-a.e.}$$

Hence, for all s outside a set of μ-measure zero there exists $M_s \subset \Omega$ with $P(M_s) = 1$ such that

$$G_{k_l}(s, \omega) G_{k_l}(t, \omega) \to G(s, \omega) G(t, \omega) \quad \text{for } \omega \in M_s \cap M_t.$$

Since these r.v.'s are μ-a.e. Gaussian, they are integrable enough to conclude that for all s, t outside a set of μ-measure zero, $EG_{k_l}(s)G_{k_l}(t) \to EG(s)G(t)$, or

$$\int_{L_p} x(s)x(t) \, d\gamma_{k_l}(x) \to \int_{L_p} x(s)x(t) \, d\gamma(x). \tag{12}$$

Using (6) and the fact that the $U_k(Z)$ are uniformly bounded, we obtain, by the same argument, that for some subsequence $\{k_{l_m}\}$ of $\{k_l\}$ and for all s and t outside a set of μ-measure zero,

$$EU_{k_{l_m}} Z(s) U_{k_{l_m}} Z(t) \to EZ(s)Z(t). \tag{13}$$

Now (4) follows from (9), (12) and (13). □

Now we can prove the final result.

THEOREM. (a) *Let X be a centered $L_p(T, \Sigma, \mu)$-valued r.v., $0 < p < 1$, and let X_i, $i \in \mathbf{N}$, be independent copies of X. Then, if*

$$\int_T \left(EX^2(t)\right)^{p/2} d\mu(t) < \infty, \tag{14}$$

we have

$$\mathfrak{L}\left(\sum_{i=1}^{n} X_i / n^{1/2} \right) \to_w \gamma, \tag{15}$$

where γ is the symmetric Gaussian p.m. determined by the covariance of X, i.e. such that

$$\int_{L_2} x(s)x(t)\,d\gamma(x) = EX(s)X(t) \tag{16}$$

for all s and t outside a set of μ-measure zero. (b) *Conversely, if X is a L_p-valued r.v. such that $\{\mathfrak{L}(\sum_{i=1}^n X_i/n^{1/2})\}_{n=1}^\infty$ is weakly convergent, then X is centered and* (14) *holds* (*hence, also* (15) *and* (16)).

PROOF. For the direct part, let $T_r \uparrow T$, $r \in \mathbf{N}$, with $T_r \in \Sigma$ and $\mu(T_r) < \infty$. For each T_r and for each $c > 0$, consider

$$X_c^{(r)}(t,\omega) = X(t,\omega)I_{[|X(t,\omega)|\leqslant c]}(t,\omega)I_{T_r}(t), \tag{17}$$

and similarly define $X_{i,c}^{(r)}$ from the r.v.'s X_i. Set also $S_n = \sum_{i=1}^n X_i$ and $S_{n,c}^{(r)} = \sum_{i=1}^n X_{i,c}^{(r)}$. Then,

(18)

$$E\|S_n/n^{1/2} - (S_{n,c}^{(r)} - ES_{n,c}^{(r)})/n^{1/2}\|_p = n^{-p/2}E\|(S_n - S_{n,c}^{(r)}) - E(S_n - S_{n,c}^{(r)})\|_p$$

$$\leqslant n^{-p/2}\int_T\left[E|(S_n - S_{n,c}^{(r)}) - E(S_n - S_{n,c}^{(r)})|^2\right]^{p/2}d\mu \leqslant \int_T\left[E\left(X - X_c^{(r)}\right)^2\right]^{p/2}d\mu$$

$$= \int_{T-T_r}\left[EX^2(t)\right]^{p/2}d\mu(t) + \int_{T_r}\left[EX^2(t)I_{[|X(t)|>c]}\right]^{p/2}d\mu(t)$$

$$\to 0 \quad \text{uniformly in } n \text{ as } r \wedge c = \min(r,c) \to \infty,$$

by condition (14). Since the variables $X_c^{(r)} - EX_c^{(r)}$ satisfy the hypotheses of Z in Lemma 5, this lemma together with (18) gives, by Lemmas 1 and 2, that

$$\mathfrak{L}\left(\sum_{i=1}^n X_i/n^{1/2}\right) \to_w \gamma \tag{15}$$

where γ is the symmetric Gaussian p.m. on L_p determined by the limit

$$\gamma = w\text{-}\lim_{r\wedge c\to\infty} \gamma_c^{(r)} \tag{19}$$

and $\gamma_c^{(r)}$ is the symmetric Gaussian law with the covariance of $X_c^{(r)}$ μ-a.e. Next we prove (16). Let $X(t,\omega)$ be a (jointly measurable) version of X, and let $\{r_l\}$, $\{c_l\}$ be sequences of integers increasing to $+\infty$. Then

$$|X_{c_l}^{(r_l)}(s,\omega)X_{c_l}^{(r_l)}(t,\omega)| \leqslant |X(s,\omega)X(t,\omega)|,$$

and since $E|X(s)X(t)| \leqslant (EX^2(s))^{1/2}(EX^2(t))^{1/2} < \infty$ except for s and t in a set T_0' of μ-measure zero (by condition (14)), it follows that

$$EX_{c_l}^{(r_l)}(s)X_{c_l}^{(r_l)}(t) \to EX(s)X(t), \qquad s,t \notin T_0', \tag{20}$$

by dominated convergence. On the other hand, the argument leading to (12) in the proof of Lemma 5 gives now that there exists a subsequence $\{l_m\}$ and a set T_0'' of μ-measure zero such that

$$\int_{L_p} x(s)x(t)\,d\gamma_{c_{l_m}}^{(r_{l_m})}(x) \to \int_{L_p} x(s)x(t)\,d\gamma(x), \qquad s,t \notin T_0''. \tag{21}$$

By Lemma 5, the covariances of $\gamma_{c_l}^{(r_l)}$ and $X_{c_l}^{(r_l)}$ coincide μ-a.e. Then, (16) follows from this, and from (20) and (21). Part (a) is thus proved.

Next we prove part (b). Let X' be an independent copy of X, and let $\tilde{X} = X - X'$. Then $\{\mathfrak{L}(\sum_{i=1}^{n} \tilde{X}_i / n^{1/2})\}_{n=1}^{\infty}$ is weakly convergent. Let

$$\delta_{n,t} = \inf\left[u : nP\left\{|\tilde{X}(t)| > u\right\} \leqslant 1/8 \cdot 3^p\right].$$

The sets $\{\omega : |\tilde{X}(t, \omega)| > \delta_{n,t}\}$ decrease to a P-null set as $n \uparrow \infty$ and therefore, $E\tilde{X}^2(t) I_{[|\tilde{X}(t)| \leqslant \delta_{n,t}]} \uparrow E\tilde{X}^2(t) \leqslant \infty$ μ-a.e. Hence, applying Lemmas 3 and 4 we have

$$\begin{aligned}
\infty > \lim_n E\left\| \sum_{i=1}^{n} \tilde{X}_i / n^{1/2} \right\|_p &\geqslant \int_T \varliminf_n n^{-p/2} E\left| \sum_{i=1}^{n} \tilde{X}_i(t) \right|^p d\mu(t) \\
&\geqslant C^{-1} \int_T \varliminf_n \left[E\tilde{X}^2(t) I_{[|\tilde{X}(t)| \leqslant \delta_{n,t}]} \right]^{p/2} d\mu(t) \\
&= C^{-1} \int_T \left[E\tilde{X}^2(t) \right]^{p/2} d\mu(t).
\end{aligned}$$

So, we have that $E\tilde{X}^2(t) < \infty$ μ-a.e. and

$$\int_T \left[E(X(t) - EX(t))^2 \right]^{p/2} d\mu(t) = 2^{-p/2} \int_T \left[E\tilde{X}^2(t) \right]^{p/2} d\mu(t) < \infty. \tag{22}$$

In particular,

$$E \int_T |X(t) - EX(t)|^p \, d\mu(t) \leqslant \int_T \left[E(X(t) - EX(t))^2 \right]^{p/2} d\mu(t) < \infty,$$

and therefore the process $X(t) - EX(t)$ has almost all its trajectories in $L_p(T, \Sigma, \mu)$. So the function (class of functions) $EX(t)$ is in L_p. On the hand, by the first part of this theorem and (22), it follows that $X(t) - EX(t)$ also satisfies the central limit theorem, but this implies that the functions $n^{1/2} EX(t)$ are L_p bounded. Hence $EX(t) = 0$ μ-a.e. In particular, (22) becomes (14) and part (b) is proved. □

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