PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 1, May 1983

ON THE EXISTENCE OF EQUIVARIANT EMBEDDINGS OF PRINCIPAL BUNDLES INTO VECTOR BUNDLES

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ABSTRACT. Let G be a finite group and let X be, say, a connected CW-complex of dimension $k \geqslant 1$. Let $\pi: E \to X$ be a principal G-bundle and $p: V \to X$ an m-dimensional G-vector-bundle with trivial action of G on X. By an equivariant embedding of π into p we understand an equivariant embedding $h: E \to V$ commuting with projections. We prove a general embedding theorem, a main special case of which is the following

THEOREM. *If $k < m$ and if the action of G on V is free outside the zero section for p, then any principal G-bundle $\pi: E \to X$ can be embedded equivariantly into $p: V \to X$.*

1. Introduction. Throughout this paper G denotes a finite group. Let $\pi: E \to X$ be a principal G-bundle, and $p: V \to X$ a real G-vector-bundle with trivial action of G on X. The actions of G on E and V are taken as right actions. By an equivariant embedding of π into p we understand a map $h: E \to V$ which commutes with projections, maps E homeomorphically onto its image $h(E)$ in V, and satisfies $h(e \cdot g) = h(e) \cdot g$ for all $e \in E$ and $g \in G$:

$$\begin{array}{ccc} E & \xrightarrow{h} & V \\ & {\scriptstyle\pi}\searrow \quad \swarrow{\scriptstyle p} & \\ & X & \end{array}$$

We shall be concerned with the existence of such equivariant embeddings. As a corollary to our main theorem we get the following

THEOREM (COROLLARY 4.2). *Suppose that X is a connected, topological space with a nondegenerate base point and with the homotopy type of a CW-complex of dimension $k \geqslant 1$, and let $p: V \to X$ be an m-dimensional G-vector-bundle. If $k < m$ and if the action of G on V is free outside the zero section for p, then any principal G-bundle $\pi: E \to X$ can be embedded equivariantly into the G-vector-bundle $p: V \to X$.*

The study of equivariant embedding theorems is motivated by the embedding theorem for finite covering maps $\pi: E \to X$ into trivial vector bundles of dimension $m > k$ proved by the first author [**4**], and generalized to arbitrary vector bundles of dimension $m > k$ by Duvall and Husch [**2**]. For 2-fold coverings, embedding theorems into trivial vector bundles in the unstable dimensions $m \leqslant k$ have been obtained by Prevot [**6**].

Received by the editors August 12, 1981 and, in revised form, August 28, 1982.
1980 *Mathematics Subject Classification*. Primary 57M12; Secondary 55R25, 57S17.
Key words and phrases. Principal G-bundle, G-vector-bundle, equivariant embeddings.

The authors are grateful to Stefan Waner for reading the first version of this paper very carefully and for making detailed suggestions, which led to considerable improvements and simplifications in the exposition.

2. Preliminaries. Let p: $V \to X$ be a fixed real G-vector-bundle. Normally, a G-vector-bundle is just an ordinary locally trivial vector bundle p: $V \to X$ with G-actions on the total space V and the base space X, such that p is equivariant, and such that G acts linearly in the fibres of p. In addition, we shall assume throughout that the G-action on X is trivial, and that the action of G is effective in each fibre of p: $V \to X$. We note that p admits G-equivariant local trivializations. On request, Karsten Grove informs us that this can be proved with small amendments to the proof in his paper [**3**].

For each integer $k \geq 1$, the Whitney sum bundle

$$p^k\colon V^k = V \oplus \cdots \oplus V \to X$$

has an induced G-vector-bundle structure by coordinatewise action of G. Viewing p^k as a subbundle of p^{k+1} by putting the zero vector in the last coordinate and then taking the direct limit we obtain the infinite Whitney sum bundle p^∞: $V^\infty \to X$. By definition, an element $v = (v_i)_{i=1}^\infty$ in V^∞ is a sequence of vectors in the same fibre of p: $V \to X$ all but a finite number of which are the zero vector. Also p^∞: $V^\infty \to X$ has an induced G-vector-bundle structure by coordinatewise action of G.

For any G-space Z, and any element $g \neq 1$ in G, we denote by $Z^{[g]}$ the set of points in Z kept fixed by g. By Z_G we denote the subset of Z defined by

$$Z_G = Z \setminus \bigcup_{g \neq 1} Z^{[g]}.$$

Clearly, the action of G on Z induces a free G-action on Z_G.

We shall in particular consider the spaces V_G and V_G^∞. Since p: $V \to X$ is G-locally-trivial, it is easy to prove that the induced projections

$$p_G\colon V_G \to X \quad \text{and} \quad p_G^\infty\colon V_G^\infty \to X$$

are locally trivial fibrations with free G-actions on the fibres.

LEMMA 2.1. *The fibration p_G^∞: $V_G^\infty \to X$ has contractible fibres.*

PROOF. A fibre of p_G^∞ is homeomorphic to a space

$$R_G^\infty = \varinjlim_k (R^m)_G^k,$$

where R^m denotes euclidean m-space equipped with a certain effective G-action by linear isomorphisms.

Since the G-action on $(R^m)^k$ is coordinatewise, we get an identification

$$(R^m)_G^k = (R^m)^k \setminus \bigcup_{g \neq 1} (\mathrm{Fix}(g))^k,$$

where $\mathrm{Fix}(g)$ denotes the fixpoint set for the isomorphism g: $R^m \to R^m$ defined by the element $g \in G$.

Since $g \neq 1$ and the action of G on R^m is effective, Fix(g) is a subspace of codimension $\geqslant 1$ in R^m. Hence $(\mathrm{Fix}(g))^k$ has codimension $\geqslant k$ in $(R^m)^k = R^{mk}$ for each $g \neq 1$.

A simple transversality argument shows now that $(R^m)^k_G$ is $(k-2)$-connected. Hence the direct limit of these spaces, R^∞_G, is contractible, since it has the homotopy type of a CW-complex. This proves the lemma. □

3. Equivariant embeddings into p^∞_G. Let ω: $EG \to BG$ be the universal numerable principal G-bundle constructed by Milnor. We follow the exposition in Husemoller [**5**]. By definition the elements of the total space EG are equivalence classes of sequences

$$\langle g, t\rangle = (t_0 g_0, t_1 g_1, t_2 g_2, \ldots),$$

where $g_j \in G$ and $t_j \in I = [0,1]$ such that only a finite number of $t_j \neq 0$ and $\sum_{j=0}^\infty t_j = 1$.

PROPOSITION 3.1. *There exists a continuous map*

$$\mu\colon V^\infty_G \times EG \to V^\infty_G$$

with the following properties:

(i) $p^\infty_G \circ \mu(v, \langle g, t\rangle) = p^\infty_G(v)$,

(ii) $\mu(v, \langle g, t\rangle \cdot g') = \mu(v, \langle g, t\rangle)\cdot g'$, *for* $v \in V^\infty_G$, $\langle g, t\rangle \in EG$ *and* $g' \in G$.

We think of μ *as a fibrewise action of* EG *on* V^∞_G.

PROOF. Define μ: $V^\infty_G \times EG \to V^\infty_G$ by

$$\mu(v, \langle g, t\rangle) = \left(\left(t_{k-i} v_i \cdot g_{k-i}\right)_{i=1}^k\right)_{k=1}^\infty,$$

and check that it has the properties (i) and (ii). □

We are now ready to prove our first main result.

THEOREM 3.2. *Any principal G-bundle* π: $E \to X$ *embeds equivariantly into* p^∞_G: $V^\infty_G \to X$ *through a fibrewise map.*

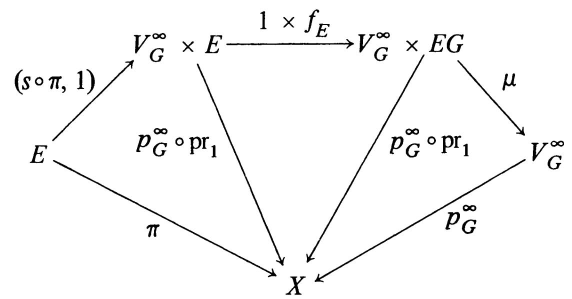
PROOF. Let s: $X \to V^\infty_G$ be a section of p^∞_G: $V^\infty_G \to X$. Such a section exists by Dold [**1**], since the fibres of p^∞_G are contractible by Lemma 2.1. Let f_E: $E \to EG$ be the map between total spaces in the classifying map for π:

$$\begin{array}{ccc} E & \xrightarrow{f_E} & EG \\ \pi\downarrow & & \downarrow\omega \\ X & \xrightarrow[f_B]{} & BG \end{array}$$

An explicit equivariant fibrewise embedding as required can then be constructed as the composition

$$h = \mu \circ (1 \times f_E) \circ (s \circ \pi, 1)$$

of maps in the diagram



where pr_1 denotes projection onto the first factor. Here, G acts on $V_G^\infty \times E$ and on $V_G^\infty \times EG$ via the second factor. That h is G-equivariant now follows from Proposition 3.1(ii). □

4. Equivariant embeddings into p. Suppose that X is a connected, topological space with a nondegenerate base point and with the homotopy type of a CW-complex of dimension $k \geqslant 1$. Let $p\colon V \to X$ be an m-dimensional G-vector-bundle in which G acts effectively on each fibre of p.

Denote by

$$m(p, G) = \min_{g \neq 1} \operatorname{codim} \operatorname{Fix}(g),$$

the minimum of the codimensions of the fixpoint sets $\operatorname{Fix}(g)$ for the isomorphisms $g\colon R^m \to R^m$ defined in an arbitrary fibre R^m of p by the elements $g \in G$, $g \neq 1$.

Since the action of G in each fibre of p is effective we have $1 \leqslant m(p, G) \leqslant m$.

We are now ready to prove our main theorem.

THEOREM 4.1. *With notation as above suppose that* $1 \leqslant k < m(p, G)$. *Then any principal G-bundle* $\pi\colon E \to X$ *can be embedded equivariantly into the G-vector-bundle* $p\colon V \to X$.

As a corollary to Theorem 4.1 we get immediately the theorem stated in the introduction.

COROLLARY 4.2. *With notation as above suppose that* $1 \leqslant k < m$. *If the action of G is free outside the zero section for p, then any principal G-bundle* $\pi\colon E \to X$ *can be embedded equivariantly into the G-vector-bundle* $p\colon V \to X$.

For the proof of Theorem 4.1 we need two lemmas. The first lemma can be proved along the same lines as G. W. Whitehead [7, Chapter II, §3, Lemma 3.1, p. 70].

LEMMA 4.3. *Let* $F' \xrightarrow{i'} E' \xrightarrow{p'} B$ *be a subfibration of the fibration* $F \xrightarrow{i} E \xrightarrow{p} B$ *over the same base space B. Suppose that the pair* (F, F') *is n-connected and that B is a CW-complex of dimension* $\leqslant n$. *Then any section s in p is vertical homotopic to a section* s' *in* p'.

LEMMA 4.4. *Suppose that* $2 \leq m(p, G)$. *Consider* $p_G \colon V_G \to X$ *as a subfibration of* $p_G^\infty \colon V_G^\infty \to X$ *by inclusion on the first coordinate. Then the pair of fibres for the pair of fibrations* (p_G^∞, p_G) *is* $(m(p, G) - 1)$*-connected.*

PROOF OF LEMMA 4.4. Following the notation from the proof of Lemma 2.1, a fibre in $P_G \colon V_G \to X$ can be identified with

$$R^m \setminus \bigcup_{g \neq 1} \mathrm{Fix}(g).$$

By assumption $\operatorname{codim} \mathrm{Fix}(g) \geq m(p, G) \geq 2$ for each $g \in G$, $g \neq 1$. Hence by a simple transversality argument the fibres in p_G are $(m(p, G) - 2)$-connected.

Since by Lemma 2.1 the fibres of p_G^∞ are contractible, the pair of fibres for the pair of fibrations (p_G^∞, p_G) is clearly $(m(p, G) - 1)$-connected. □

PROOF OF THEOREM 4.1. Let $\pi \colon E \to X$ be an arbitrary principal G-bundle. We shall apply the usual technique for transforming problems of bundle maps into section problems. Let therefore $\mathrm{Emb}(\pi, p_G)$, respectively $\mathrm{Emb}(\pi, p_G^\infty)$, denote the fibration over X for which the sections are the equivariant fibrewise embeddings of π into p_G, respectively p_G^∞. In the obvious way, we consider $\mathrm{Emb}(\pi, p_G)$ as a subfibration of $\mathrm{Emb}(\pi, p_G^\infty)$. Since an equivariant embedding of a principal G-bundle is completely determined by its values on a single element in each fibre, the pair of fibres for the pair of fibrations $(\mathrm{Emb}(\pi, p_G^\infty), \mathrm{Emb}(\pi, p_G))$ can be identified with the pair of fibres for the pair of fibrations (p_G^∞, p_G), and is therefore $(m(p, G) - 1)$-connected by Lemma 4.4. By Lemma 4.3, an equivariant embedding of π into p_G^∞ represented by a section in $\mathrm{Emb}(\pi, p_G^\infty)$ can therefore be deformed into an equivariant embedding of π into p_G represented by a section in $\mathrm{Emb}(\pi, p_G)$. In particular we obtain an equivariant embedding of π into p, and Theorem 4.1 is proved. □

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