PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 1, May 1983

A SPACE WHICH CONTAINS NO REALCOMPACT DENSE SUBSPACE

TOSHIJI TERADA

ABSTRACT. A Tychonoff space which contains no realcompact space as a dense subspace is constructed. Let $\mathfrak{D}\mathfrak{R}$ be the class of all spaces which contain some realcompact spaces as dense subspaces. Then, as a consequence of the above result, it follows that $\mathfrak{D}\mathfrak{R}$ is not closed-hereditary.

All spaces considered here are Tychonoff topological spaces. Let $\mathfrak{D}\mathfrak{R}$ be the class of all spaces which contain some realcompact spaces as dense subspaces. In [**5**], the author noticed that the class $\mathfrak{D}\mathfrak{R}$ is closed-hereditary if and only if $\mathfrak{D}\mathfrak{R}$ is the class of all spaces. If we assume the existence of a measurable cardinal, then each discrete space of a measurable cardinal cannot contain a realcompact space as a dense subspace, and hence $\mathfrak{D}\mathfrak{R}$ is not closed-hereditary. However, without any additional set theoretical assumption, it is not clear whether $\mathfrak{D}\mathfrak{R}$ is closed-hereditary.

In this note we shall give naively a space which contains no realcompact space as a dense subspace. Hence it follows that $\mathfrak{D}\mathfrak{R}$ is not closed-hereditary.

A space is called Oz (or perfectly κ-normal) if every regular closed subset is a zero-set (see [**2, 3, 4**]). If X is Oz, then every dense subspace is z-embedded in X. Ščepin showed that every product of metrizable spaces is Oz. Blair and Hager [**1**] showed that a z-embedded subspace Y of a realcompact space X is realcompact if and only if Y is G_δ-closed in X (i.e. for any point x in $X - Y$ there is a G_δ-set S of X such that $x \in S$ and $S \cap Y = \varnothing$). From this fact, the following lemma is obvious.

LEMMA 1. *Let Y be a dense subspace of a realcompact Oz-space X. Then the following are equivalent.*

(1) *Y is realcompact.*

(2) *Y is G_δ-closed in X.*

Let A be a set such that $|A| \geqslant \aleph_1$. For each α in A let X_α be the space of rational numbers. Let $p_\alpha = 0$, $q_\alpha = 1$ in X_α and let $X'_\alpha = X_\alpha - \{p_\alpha, q_\alpha\}$. The family of all nonempty finite subsets of A is denoted by $\mathfrak{F}(A)$. Let $\mathfrak{S}$ be a disjoint family of countably infinite subsets of A such that $|\mathfrak{S}| = |A|$. Since the cardinality of the set

Received by the editors April 26, 1982 and, in revised form, August 3, 1982.
1980 *Mathematics Subject Classification*. Primary 54D60, 54G20; Secondary 54B05, 54B10.
Key words and phrases. Dense subspace, realcompact, G_δ-closed, Oz, σ-product.

0002-9939/82/0000-0943/$01.50

$\bigcup\{\Pi\{X'_\beta\colon \beta \in B\}\colon B \in \mathfrak{F}(A)\}$ is equal to that of A, there is a one-to-one map t: $\bigcup\{\Pi\{X'_\beta\colon \beta \in B\}\colon B \in \mathfrak{F}(A)\} \to \mathfrak{S}$. For each B in $\mathfrak{F}(A)$ we define a map h_B: $\Pi\{X'_\beta\colon \beta \in B\} \to \Pi\{X_\alpha\colon \alpha \in A\}$ in the following way;

$$\pi_\alpha\big(h_B(\langle y_\beta\rangle)\big) = \begin{cases} y_\alpha & \text{if } \alpha \in B, \\ q_\alpha & \text{if } \alpha \in t(\langle y_\beta\rangle) - B, \\ p_\alpha & \text{otherwise,} \end{cases}$$

where π_α: $\Pi\{X_\alpha\colon \alpha \in A\} \to X_\alpha$ is the natural projection and $\langle y_\beta\rangle$ means a member of $\Pi\{X'_\beta\colon \beta \in B\}$ whose β-coordinate is y_β. Let Z_A be the subspace of $\Pi\{X_\alpha\colon \alpha \in A\}$ defined by $\bigcup\{h_B(\Pi\{X'_\beta\colon \beta \in B\})\colon B \in \mathfrak{F}(A)\}$. Obviously Z_A is dense in $\Pi\{X_\alpha\colon \alpha \in A\}$. In the rest of this note, for the sake of simplicity, $\Pi\{X_\alpha\colon \alpha \in A\}$ is denoted by X_A. If $\langle x_\alpha\rangle = h_B(\langle y_\beta\rangle)$, then $\langle y_\beta\rangle$ is denoted by $s(\langle x_\alpha\rangle)$ and B is denoted by $i(\langle x_\alpha\rangle)$. Countable intersections of canonical open subsets of X_A are called canonical G_δ-sets.

Lemma 2. *The density of Z_A is not less than $\aleph_1$.*

Proof. Assume that there is a dense subspace Y in Z_A such that $|Y| \leq \aleph_0$. Note that Y is also dense in X_A. For each point $\langle x_\alpha\rangle$ in Y the set $\{\alpha \in A\colon x_\alpha \neq p_\alpha\}$ is countable. Let $A_{\langle x_\alpha\rangle} = \{\alpha \in A\colon x_\alpha \neq p_\alpha\}$. Then the cardinality of the set $A' = \bigcup\{A_{\langle x_\alpha\rangle}\colon \langle x_\alpha\rangle \in Y\}$ is not more than $\aleph_0$. Hence there is a member α_0 in $A - A'$. This implies that the α_0-coordinate of every member of Y is p_{α_0}. But this shows that Y is not dense in X_A. This is a contradiction.

Proof. *Z_A contains no realcompact space as a dense subspace.*

Proof. Assume that Z_A contains a realcompact space Y as a dense subspace. Since Y is dense in the realcompact Oz-space X_A, Y must be G_δ-closed in X_A by Lemma 1. Let σ_A be the σ-product of $\{X_\alpha\colon \alpha \in A\}$ with the base point $\langle p_\alpha\rangle$. Notice that $\sigma_A \cap Z_A = \varnothing$. Let $\langle x_\alpha\rangle$ be an arbitrary point of σ_A. Then there is a canonical G_δ-set $S_{\langle x_\alpha\rangle}$ in X_A such that $\langle x_\alpha\rangle \in S_{\langle x_\alpha\rangle}$ and $S_{\langle x_\alpha\rangle} \cap Y = \varnothing$. We can assume that $S_{\langle x_\alpha\rangle} = \Pi\{S_{x_\alpha}\colon \alpha \in A\}$ where S_{x_α} is a G_δ-set in X_α and $S_{x_\alpha} = X_\alpha$ except countably many α's. Let $B_{\langle x_\alpha\rangle} = \{\alpha \in A\colon x_\alpha \neq p_\alpha\}$ and let $C_{\langle x_\alpha\rangle} = \{\alpha \in A\colon S_{x_\alpha} \neq X_\alpha\}$. Note that $B_{\langle x_\alpha\rangle}$ is finite and $C_{\langle x_\alpha\rangle}$ is countable. Now, we construct a sequence $\{C_n\colon n = 0, 1, 2, \ldots\}$ of subsets of A in the following way. Let C_0 be an arbitrary nonempty countable subset of A. Assume that C_n is constructed such that $|C_n| \leq \aleph_0$. Let $\sigma_A^n = \{\langle x_\alpha\rangle \in \sigma_A\colon B_{\langle x_\alpha\rangle} \subset C_n\}$. Then the cardinality of σ_A^n is at most $\aleph_0$. Let $C_{n+1} = C_n \cup \bigcup\{C_{\langle x_\alpha\rangle}\colon \langle x_\alpha\rangle \in \sigma_A^n\}$. It is obvious that $|C_{n+1}| \leq \aleph_0$. Let $C = \bigcup\{C_n\colon n = 0, 1, 2, \ldots\}$. Then obviously $|C| \leq \aleph_0$. We shall show that $t(s(\langle x_\alpha\rangle)) \cap C \neq \varnothing$ for each element $\langle x_\alpha\rangle$ of Y. Assume that $t(s(\langle x_\alpha\rangle)) \cap C = \varnothing$. Since $i(\langle x_\alpha\rangle)$ is finite, $i(\langle x_\alpha\rangle) \cap C \subset C_n$ for some n. Let $\langle z_\alpha\rangle$ be the element of σ_A determined in the following way:

$$z_\alpha = \begin{cases} x_\alpha & \text{if } \alpha \in i(\langle x_\alpha\rangle) \cap C, \\ p_\alpha & \text{otherwise.} \end{cases}$$

Then $\langle z_\alpha \rangle \in \sigma_A^n$, and hence $C_{\langle z_\alpha \rangle} \subset C_{n+1}$. Further, since $x_\alpha = z_\alpha$ for each α in $C_{\langle z_\alpha \rangle}$, it follows that $\langle x_\alpha \rangle \in S_{\langle z_\alpha \rangle}$. But this is a contradiction since $S_{\langle z_\alpha \rangle} \cap Y = \varnothing$. Hence it is proved that $t(s(\langle x_\alpha \rangle)) \cap C \neq \varnothing$ for each $\langle x_\alpha \rangle$ in Y. Then, since $\mathfrak{S}$ is a disjoint family of countably infinite subsets of A, the cardinality of Y must be at most $\aleph_0$. But this contradicts Lemma 2.

Since $|Z_A| \leq |\mathfrak{F}(A)| \cdot \sup\{|\Pi\{X'_\beta: \beta \in B\}| : B \in \mathfrak{F}(A)\} = |A| \cdot \aleph_0$, if $|A| = \aleph_1$, then $|Z_A| \leq \aleph_1$. Further, the cardinality of every nonrealcompact space is not less than $\aleph_1$. Hence we have the following conclusion.

THEOREM. *There is a space X with the following properties*:
(1) $|X| = \aleph_1$,
(2) *every dense subspace of X is not realcompact.*

COROLLARY. $\mathfrak{D}\mathfrak{R}$ *is not closed-hereditary.*

REFERENCES

1. R. L. Blair and A. W. Hager, *Extensions of zero-sets and of real-valued functions*, Math. Z. **136** (1974), 41–52.

2. R. L. Blair, *Spaces in which special sets are z-embedded*, Canad. J. Math. **28** (1976), 673–690.

3. E. V. Ščepin, *On topological products, groups, and a new class of spaces more general than metric ones*, Dokl. Akad. Nauk SSSR **226** (1976), 527–529.

4. T. Terada, *Note on z-, C*-, and C-embedding*, Sci. Rep. Tokyo Kyoiku Daigaku Sect. A **13** (1975), 129–132.

5. ______, *Dense subspaces of topological spaces* (submitted).

DEPARTMENT OF MATHEMATICS, FACULTY OF ENGINEERING, YOKOHAMA NATIONAL UNIVERSITY, 156, HODOGAYA, YOKOHAMA, JAPAN