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CHARACTERIZATIONS OF $\mathcal{F}$-FIBRATIONS

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ABSTRACT. We obtain a reformulation of May's notion of an "$\mathcal{F}$-fibration" and use it to sharpen results of Booth, Heath and Piccinini on the characterization of universal $\mathcal{F}$-fibrations.

We are concerned with the classification of various types of fibrations. A general framework for this study was provided by May (see [**3**, §1–4]). As there, we assume that a category of fibres $\mathcal{F}$ is given. This is just a suitable category of spaces in which the fibres of fibrations are constrained to lie. Throughout all spaces are to be compactly generated and weak Hausdorff. An $\mathcal{F}$-space is a map $p\colon E \to B$ such that each fibre $E_b = p^{-1}(b)$ is an object of $\mathcal{F}$. An $\mathcal{F}$-map $q \to p$ is a commutative diagram

$$\begin{array}{ccc} D & \xrightarrow{g} & E \\ \downarrow q & & \downarrow p \\ A & \xrightarrow{f} & B \end{array}$$

such that $g\colon D_a \to E_{f(a)}$ is a morphism of $\mathcal{F}$ for each $a \in A$. An $\mathcal{F}$-map with domain $q \times 1\colon D \times I \to A \times I$ is an $\mathcal{F}$-homotopy and a homotopy equivalence relative to $\mathcal{F}$ is an $\mathcal{F}$-homotopy equivalence. For purposes of classification we assume that $\mathcal{F}$ contains a distinguished object F such that $\mathcal{F}(F, X)$ is nonempty for all $X \in \mathcal{F}$, that every morphism of $\mathcal{F}$ is a weak homotopy equivalence, and that composition with ϕ,

$$\mathcal{F}(1, \phi)\colon \mathcal{F}(F, F) \to \mathcal{F}(F, X),$$

is a weak homotopy equivalence for each $\phi \in \mathcal{F}(F, X)$.

An $\mathcal{F}$-space $p\colon E \to B$ is an $\mathcal{F}$-fibration if it has the $\mathcal{F}$-covering homotopy property, abbreviated $\mathcal{F}$-CHP, with respect to all $\mathcal{F}$-spaces. That is, for every $\mathcal{F}$-space $q\colon D \to A$, every $\mathcal{F}$-map $(g, f)\colon q \to p$ and every homotopy h of f, there exists a homotopy H of g such that (H, h) is an $\mathcal{F}$-homotopy. We say that p is a Serre $\mathcal{F}$-fibration if it has the $\mathcal{F}$-CHP with respect to all $\mathcal{F}$-spaces with base space a CW-complex. Since (g, f) necessarily factors through the induced $\mathcal{F}$-space $f^*(p)\colon f^*(E) \to A$, it is clear that the $\mathcal{F}$-CHP will hold in general if it holds whenever q is an induced $\mathcal{F}$-space.

Booth, Heath and Piccinini [**1, 2**] introduced the following construction. Let $q\colon D \to A$ and $p\colon E \to B$ be $\mathcal{F}$-spaces. Define $D * E = \bigcup_{(a,b)\in A\times B}\mathcal{F}(D_a, E_b)$ and let $q * p\colon D * E \to A \times B$ denote the obvious projection map. With a suitable topology

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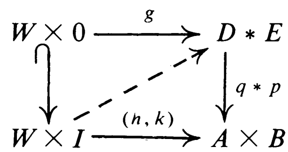
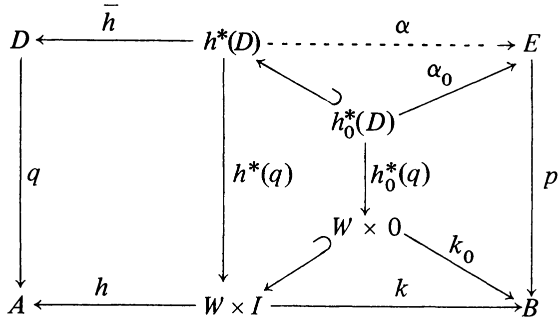
on $D * E$, $q * p$ is a $\mathcal{G}$-space, where $\mathcal{G}$ is the associated principal category of fibres of $\mathcal{F}$ (see [**3**, 4.3]). When $D = F$ and $A = *$, $q * p$ is the associated principal $\mathcal{G}$-space of p. We have the following generalization of [**3**, 4.6 and **2**, Corollary 7].

Proposition 1. *If q and p are $\mathcal{F}$-fibrations, then $q * p$ is a (Hurewicz) fibration.*

Proof. The existence of a relative lifting for the diagram



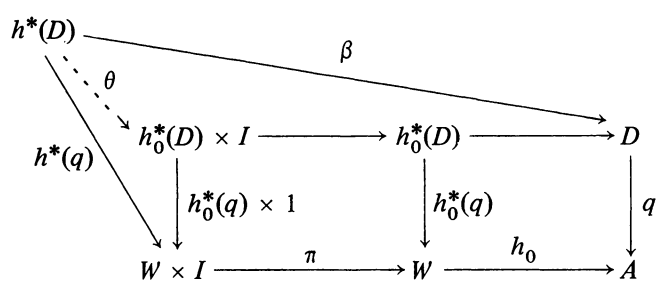
is equivalent by [**1**, Lemma 1.2] to the existence of an $\mathcal{F}$-map completing the diagram



where $\alpha_0(w, d) = g(w, 0)(d)$. Since p is an $\mathcal{F}$-fibration there exists an $\mathcal{F}$-homotopy (K, k): $h_0^*(q) \times 1 \to p$ of (α_0, k_0). Define a homotopy f: $W \times I \times I \to A$ of h by

$$f(w, t, s) = \begin{cases} h(w, t - s), & t - s \geq 0, \\ h(w, 0), & t - s \leq 0. \end{cases}$$

Since q is an $\mathcal{F}$-fibration there exists a homotopy F: $h^*(D) \times I \to D$ of $\bar{h}$ such that (F, f): $h^*(q) \times 1 \to q$ is an $\mathcal{F}$-homotopy. Define β: $h^*(D) \to D$ by the rule $\beta(w, t, d) = F(w, t, d, t)$, where $q(d) = h(w, t)$. This gives rise to the commutative diagram



where the unique $\mathcal{F}$-map θ has the property that $\theta \mid h_0^*(D) = 1$. Set $\alpha = K \cdot \theta$.

This has the following consequence.

PROPOSITION 2. *An $\mathcal{F}$-space $p\colon E \to B$ is an $\mathcal{F}$-fibration if, and only if, $p * p$ is a fibration.*

PROOF. Assume $p * p$ is a fibration. Given a space W and a map $f\colon W \to B$ we must show that, for every $\mathcal{F}$-map $(h, k)\colon f^*(p) \to p$ and every homotopy $K\colon W \times I \to B$ of k, there exists a homotopy $H\colon f^*(E) \times I \to E$ of h such that $(H, K)\colon f^*(p) \times 1 \to p$ is an $\mathcal{F}$-map. But, by [**1**, Lemma 1.2], this is equivalent to the existence of a relative lifting for the diagram

$$\begin{array}{ccc} W \times 0 & \xrightarrow{\quad g \quad} & E * E \\ \cap\big\downarrow & \nearrow & \big\downarrow p * p \\ W \times I & \xrightarrow{(f \cdot \pi,\, K)} & B \times B \end{array}$$

where $g(w, 0)(y) = h(w, y)$, $w \in W$, $y \in E$.

REMARKS. (i) Using the same proofs the previous results remain valid with Serre $\mathcal{F}$-fibrations in the hypotheses and Serre fibrations in the conclusions.

(ii) The notion of an admissible category of $\mathcal{F}$-spaces was introduced in [**1**] as a full subcategory $\mathcal{A}$ of the category of all $\mathcal{F}$-spaces such that the following properties hold.

(1) The projection $F \to *$ is an object of $\mathcal{A}$.

(2) If $p\colon E \to B$ is $\mathcal{F}$-homeomorphic over B to an object of $\mathcal{A}$, then p is an object of $\mathcal{A}$.

(3) If $p\colon E \to B$ is an object of $\mathcal{A}$ and $f\colon A \to B$ is a map, then $f^*(p)\colon F^*(E) \to A$ is an object of $\mathcal{A}$.

(4) If p and q are objects of $\mathcal{A}$, then $p * q$ is a fibration.

Clearly (1)–(3) are just minimal closure conditions and the substantive axiom is (4). By Propositions 1 and 2, this axiom implies and is implied by the simple assumption that all objects of $\mathcal{A}$ are $\mathcal{F}$-fibrations. This makes it a very easy matter to verify that the usual categories of interest are admissible. Actually [**1**] restricts all base spaces to be CW-complexes and only requires $p * q$ to be a Serre fibration in (4), but the same comment still applies.

Let $\mathcal{E}\mathcal{F}(A)$ denote the set of equivalence classes of $\mathcal{F}$-fibrations over A under the equivalence relation generated by the $\mathcal{F}$-maps over A (we could restrict to some admissible full subcategory of $\mathcal{F}$-fibrations). In cases where the morphisms of $\mathcal{F}$ are $\mathcal{F}$-homotopy equivalences, this equivalence relation coincides, over good base spaces (e.g. CW-complexes), with the more restrictive notion of $\mathcal{F}$-homotopy equivalence (see [**3**, 2.6]). Since homotopic maps induce equivalent $\mathcal{F}$-fibrations [**3**, 2.5], $\mathcal{E}\mathcal{F}$ defines a contravariant set-valued functor on the homotopy category of CW-complexes. Any $\mathcal{F}$-fibration $p\colon E \to B$ determines a natural transformation

$$\theta\colon [?, B] \to \mathcal{E}\mathcal{F}(?).$$

Here B need not be a CW-complex and unbased homotopy classes are understood. We have the following characterization of universal $\mathcal{F}$-fibrations.

THEOREM 3. *Consider the following conditions on an $\mathcal{F}$-fibration $p: E \to B$.*

(1) θ: $[A, B] \to \mathcal{E}\mathcal{F}(A)$ *is a bijection for all CW-complexes A.*

(2) *The based analog of condition* (1) *holds.*

(3) *The associated principal fibration of p has aspherical total space* $F * E$.

(4) *For every CW-pair* (A, C) *and* $\mathcal{F}$*-fibration* $q: D \to A$, *any* $\mathcal{F}$*-map* $q \mid C \to p$ *extends to an* $\mathcal{F}$*-map* $q \to p$.

Assume there exists $p': E' \to B'$ *which satisfies* (3). *Then all four conditions are equivalent.*

PROOF. The implications (2) $\Rightarrow$ (1), (3) $\Rightarrow$ (2) and (3) $\Leftrightarrow$ (4) were proven in [**1**, §3]. Actually, it was assumed there that B was a CW-complex, but inspection of the proofs shows this condition to be irrelevant. It remains to prove that (1) implies (3). Thus let $p: E \to B$ satisfy (1). If $f: A \to B$ is a weak homotopy equivalence, then the induced map $F * f^*(E) \to F * E$ is a weak homotopy equivalence by the five lemma. Moreover, $f^*(p)$: $f^*(E) \to A$ also satisfies (1) since f induces an isomorphism on represented functors on CW-complexes. Thus, by CW-approximation, we may as well assume that B is a CW-complex and similarly for B'. Then p and p' are $\mathcal{F}$-homotopy equivalent since they both satisfy (1) and, hence, $F * E$ is homotopy equivalent to the aspherical space $F * E'$.

The question of existence of $\mathcal{F}$-fibrations with an aspherical associated principal fibration is studied in [**3**], where an explicit construction is given in favourable cases.

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