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ON FIBRE INCLUSIONS AND KÄHLER MANIFOLDS

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ABSTRACT. A map $i: X \to Y$ of CW-complexes is said to be equivalent to a fibre inclusion if there exists a fibration (up to homotopy) $X \xrightarrow{i} Y \to B$. Here some classes of maps of compact Kähler manifolds are presented which are not equivalent to a fibre inclusion.

Given a map $i: X \to Y$ of spaces of the homotopy type of connected CW-complexes, it is a classical problem, due to Massey [**9**], to decide whether this map is equivalent to a fibre inclusion, i.e. whether there exists a fibration (up to homotopy) $X \xrightarrow{i} Y \to B$.

Of course, some initial observations are obvious (e.g., the homotopy fibre of i must be a loop space). However, research on the problem has not yielded strong, easily verifiable conditions for answering this question unless the spaces involved are highly connected (cf. [**2, 10, 8**], et al.).

The aim of this note is to present some classes of maps of Kähler manifolds which are not equivalent to fibre inclusions. For the notion of Kähler manifold we refer, e.g., to [**5**]. For our purposes we only recall that the Kähler form L in $H^2(M)$, M a compact Kähler manifold of dimension $2m$, satisfies $L^m \neq 0$ in $H^*(M)$. (Cohomology is taken throughout with rational coefficients.) We note in particular that all complex Grassmann manifolds have a Kähler structure.

1. Statement of results.

THEOREM 1. *Let M denote a compact simply connected Kähler manifold with $H^2(M; \mathbf{Q}) \simeq \mathbf{Q}$. Suppose a map i: $\mathbf{C}P^n \to M$ is equivalent to a fibre inclusion. Then there is a positive integer s such that $s(n+1) = m+1$, where $m = \frac{1}{2}\dim M$.*

REMARK. For the statement of Theorem 1 it is enough to assume that $i: X \to Y$ is a map of simply connected CW-complexes with $H^*(X; \mathbf{Q}) \simeq \mathbf{Q}[x]/(x^{n+1})$ for $n \in \mathbf{N}$ and $\deg x = 2$, and with Y having the rational cohomology ring of a compact Kähler manifold with $H^2(Y; \mathbf{Q}) \simeq \mathbf{Q}$. In Theorem 1 the last mentioned condition is in fact necessary as the obvious fibre inclusion $\mathbf{C}P^n \to \mathbf{C}P^n \times \mathbf{C}P^q$, $q \geqslant 1$, shows.

COROLLARY 2. *Let $m+1$ be a prime number. Then no map i: $\mathbf{C}P^n \to M^{2m}$ is equivalent to a fibre inclusion ($n \in \mathbf{N}$ arbitrary).*

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This result, together with the above remark, may be viewed as a generalization of [7, Theorem 7.7, Parts (ii) and (iii)].

COROLLARY 3. *Let M denote a $4m$-dimensional simply connected Kähler manifold with $H^2(M; \mathbf{Q}) \simeq \mathbf{Q}$. Then no map i: $S^2 \to M$ is equivalent to a fibre inclusion.*

In case the manifold M is not $4m$-dimensional, it is more difficult to draw a corresponding conclusion. This becomes already clear from the existence of the (well-known) bundle $S^2 \to \mathbf{C}P^{2n+1} \to \mathbf{H}P^n$, $n \geqslant 1$. If we restrict attention to complex Grassmann manifolds $M \simeq U(p+q)/U(p) \times U(q)$ we have, however, for $p > 1$ (recall that then $\dim M = 2pq$).

THEOREM 4. *Let $M = U(p+q)/U(p) \times U(q)$, $p > 1$, with pq odd and $q = mp$ for $m \in \mathbf{N}$. Then no map i corresponding to an element in $\pi_2(M) \simeq \mathbf{Z}$ is equivalent to a fibre inclusion.*

Theorems 1 and 4 are related to a conjecture due to S. Halperin and R. Schultz, which may be (roughly) stated as follows: Let $X \to M \to B$ denote a compact fibration (i.e. X and B are of the homotopy type of a finite CW-complex) of $M = U(p+q)/U(p) \times U(q)$. Then for $p > 1$ and X connected, either X or B have to be contractible. For a precise statement of this conjecture, as well as a verification in the case $p = 2$, we refer to [11].

EXAMPLE. Let $X \to M \to B$ denote a compact fibration of $E = U(7)/U(3) \times U(4)$. Then either X or B is contractible. This may be shown as follows:

Consider the Poincaré polynomial $P(M)$ of $H^*(M)$:

$$P(M) = \frac{(1-u^7)(1-u^6)(1-u^5)}{(1-u)(1-u^2)(1-u^3)}, \qquad u = t^2.$$

Then there are only two possible factorings of $P(E)$ by evenly graded polynomials with nonnegative integral coefficients:

$$P(M) = \left(1 + u + \cdots + u^6\right)\left(1 + u^2 + u^3 + u^4 + u^6\right)$$

and

$$P(M) = (1 + u + u^2 + u^3 + u^4)(1 + u^2 + u^3 + u^4 + u^5 + u^6 + u^8).$$

Hence in any case $H^*(X; \mathbf{Q}) \simeq \mathbf{Q}[x]/(x^{n+1})$, $\deg x = 2$, by Proposition 5 below. As $pq + 1 = 13$ is prime, Corollary 2 (together with the remark to Theorem 1) applies. Hence for X connected, either X or B has to be contractible. This even holds for arbitrary compact X, as can be shown by the Lefshetz Fixed Point Theorem and results of [3] concerning automorphisms in the cohomology of Grassmann manifolds (for this argument see also [11]).

By a similar argument and using the signature of M one can also show that for $M = U(8)/U(4) \times U(4)$ there exists no nontrivial compact fibration $X \to M \to B$ with X connected.

I learned from R. Schultz that he has also obtained results on the nonexistence of compact fibrations of $M = U(3+q)/U(3) \times U(q)$ for certain q.

As a final statement concerning compact fibrations of Grassmann manifolds we have

PROPOSITION 5. Let $M = U(p+q)/U(p) \times U(q)$ and let $X \to M \to B$ be a compact fibration where X is simply connected. Then $H^2(X; \mathbf{Q}) \simeq \mathbf{Q}$.

2. Proofs.

PROOF OF THEOREM 1. Suppose $\mathbf{C}P^n \xrightarrow{i} M \xrightarrow{\pi} B$ is a fibration (up to homotopy). Then B is simply connected. Applying the fact [**4**] that the rational Serre spectral sequence of this fibration collapses, we obtain $H^*(M) \simeq H^*(\mathbf{C}P^n) \otimes H^*(B)$. Moreover $H^*(M)$ is a free $H^*(B)$-module. Therefore [**1**, (5.3)] implies that

$$\mathbf{Q} \to H^*(B) \xrightarrow{\pi^*} H^*(M) \to H^*(\mathbf{C}P^n) \to \mathbf{Q},$$

is coexact, i.e. $H^*(\mathbf{C}P^n)$ is isomorphic, as a $\mathbf{Q}$-algebra, to $H^*(M)$ modulo the ideal generated by $\operatorname{im} \pi^*$ (except the 0-dimensional class). Let $L \in H^2(M)$ denote the Kähler form. Then $L^m \neq 0$ in $H^*(M)$. By coexactness, L is not in $\operatorname{im} \pi^*$, but $L^r \in \operatorname{im} \pi^*$ for some r with $2 \leqslant r \leqslant n+1$. If $r \leqslant n$ there is always a $t \in \mathbf{N}$ with rt in the interval $[m-n+1, m]$. This would imply $L^{rt} = 0$, since $H^i(B) = 0$ for $i < 2m - 2n$, hence a contradiction to $L^m \neq 0$. So we have to assume $r = n+1$. To exclude the possibility that again $rt \in [m-n+1, m]$ we necessarily have $(n+1)t = m-n$ for some $t \in \mathbf{N}$, or $(n+1)(t+1) = m+1$.

PROOF OF PROPOSITION 5. By [**7**, (6.5)], the (rational) Serre spectral sequence of $X \to M \xrightarrow{\pi} B$ collapses and hence again the sequence

$$\mathbf{Q} \to H^*(B) \xrightarrow{\pi^*} H^*(M) \to H^*(X) \to \mathbf{Q}$$

is coexact. To show that $H^2(X) = \mathbf{Q}$ we suppose, on the contrary, that $L \in \operatorname{im} \pi^*$. Then $L^m \in \operatorname{im} \pi^*$, which contradicts the fact that $H^i(B) = 0$ for $i > \dim M - \dim X$. (Observe that X as the fibre of a compact fibration is a Poincaré complex and hence has a formal dimension.)

PROOF OF THEOREM 4. We again start with the coexact sequence

$$\mathbf{Q} \to H^*(B) \xrightarrow{\pi^*} H^*(M) \to H^*(S^2) \to \mathbf{Q}. \tag{1.1}$$

Moreover, it follows from the homotopy sequence and the rational homotopy of M that

$$\begin{aligned} &\pi_4(B) \otimes \mathbf{Q} \simeq \mathbf{Q} \oplus \mathbf{Q}, \qquad \pi_{2i}(B) \otimes \mathbf{Q} = \mathbf{Q} \quad \text{for } 3 \leqslant i \leqslant p, \\ &\pi_{2(q+i)-1}(B) \otimes \mathbf{Q} \simeq \mathbf{Q} \quad \text{for } 1 \leqslant i \leqslant p, \text{ and} \\ &\pi_i(B) \otimes \mathbf{Q} = 0 \quad \text{for } i \text{ otherwise.} \end{aligned}$$

By general facts about finite simply connected CW-complexes with evenly graded rational cohomology (cf. [**6**]), it follows that $H^*(B) \simeq \mathbf{Q}[d_1, d_2, d_3, \ldots, d_p]/I$, where $\deg d_1 = \deg d_2 = 4$, $\deg d_i = 2_i$, $3 \leqslant i \leqslant p$, and the ideal I is generated by relations $R_1, \ldots, R_p$ with $\deg R_j = 2(q+j)$, $1 \leqslant j \leqslant p$. The aim is now to show that under the conditions of Theorem 4 there is no cohomology algebra $H^*(B)$ with the above properties that satisfies at the same time the coexactness property (1.1). By [**3**] the cohomology of M is given by $H^*(M) \simeq \mathbf{Q}[c_1,\ldots,c_p]/J$, where $\deg c_i = 2i$, $1 \leqslant i \leqslant p$, and $J = (\overline{R}_1,\ldots,\overline{R}_p)$, where $\overline{R}_i$, $1 \leqslant i \leqslant p$, are relations in $\mathbf{Q}[c_1,\ldots,c_p]$ with $\deg \overline{R}_i = 2(q+i)$. As an additional fact, which will be important for our argument, one knows that each monomial in $\mathbf{Q}[c_1,\ldots,c_p]$ of degree $2(q+1) = 2(mp+1)$ has a nontrivial coefficient in $\overline{R}_1$. From coexactness we have that $c_1^2 \in \operatorname{im} \pi^*$. Moreover, by the same reason, $c_i \in \operatorname{im} \pi^*$ for $2 \leqslant i \leqslant p$. Therefore we may assume, without loss of generality, that

$$\begin{aligned}
\pi^*(d_1) &= \alpha c_1^2, \\
\pi^*(d_2) &= c_2 + \beta c_1^2, \qquad \alpha, \beta \in \mathbf{Q}, \\
\pi^*(d_i) &= c_i + \sum_{r \in \mathbf{N}_i^p} a_r c^r, \qquad 3 \leqslant i \leqslant p,\ a_r \in \mathbf{Q},
\end{aligned}$$

and $r \neq e_i$, the ith unit vector.

Here we use the following notation:

$\mathbf{N}^p = \mathbf{N} \times \cdots \times \mathbf{N}$ (p copies);

$c^r = c_1^{r_1} c_2^{r_2} \cdots c_p^{r_p}$; $\mathbf{N}_k^p = \{r \in \mathbf{N}^p \mid \operatorname{wt}(r) = k\}$;

$\operatorname{wt}(r) = \sum_{i=1}^p i r_i$, $r \in \mathbf{N}^p$.

Consider now $\pi^*(R_1) = t \cdot \overline{R}_1$, $t \in \mathbf{Q}$. One can easily see that for $p > 1$ the monomial $c_1 c_p^m$ does not occur in $\pi^*(\mathbf{R}_1)$. Hence $t = 0$, i.e. $\pi^*(R_1) \equiv 0$ identically. Put $R_1 = \sum_{s \in \mathbf{N}_{q+1}^p} b_s d^s$. With this notation we show that either $b_s = 0$ for all $s \in \mathbf{N}_{q+1}^p$, i.e. $R_1 \equiv 0$, or $\alpha = 0$, thus either giving a contradiction to $\pi_{2q+1}(B) \simeq \mathbf{Q}$ or to the coexactness property (1.1). For this purpose we proceed by induction according to lexicographic order of the vectors $(s_1,\ldots,s_p)$ in $d_1^{s_1} \cdots d_p^{s_p}$. The monomials $c_1^{t_1} \cdots c_p^{t_p}$ in $\pi^*(d_1^{s_1} \cdots d_p^{s_p})$ satisfy $(s_i,\ldots,s_p) \leqslant (t_1,\ldots,t_p)$. Moreover $\pi^*(d_1^{s_1} \cdots d_p^{s_p})$ has a monomial $c_1^{2s_1} \cdots c_p^{s_p}$ with coefficient α^{s_1}. Let $s = (0, s_2,\ldots,s_p)$. Then we inductively show that $b_s = 0$, as $c_2^{s_2} \cdots c_p^{s_p}$ in $\pi^*(d_2^{s_2} \cdots d_p^{s_p})$ has coefficient 1. Proceeding in the same way for $\pi^*(d_1^{s_1} \cdots d_p^{s_p})$ with $s_1 \neq 0$ we either have $b_s = 0$ or $\alpha = 0$. This proves the theorem.

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