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A NOTE ON INFINITE-DIMENSION UNDER REFINABLE MAPS

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ABSTRACT. It is shown that refinable maps preserve weak infinite-dimension, but not strong infinite-dimension.

The purpose of this note is to show that refinable maps preserve weak infinite-dimension, but not strong infinite-dimension. Under a refinable map between compacta, the domain and image must have the same finite-dimension or must both have infinite-dimension (see [**4**, Theorem 1.8(4); **6**, Theorem I, 16]).

The term compactum is used to mean a compact metric space. A map $f\colon X \to Y$ between compacta is said to be an *ε-mapping*, $\varepsilon > 0$, if $\operatorname{diam} f^{-1}(y) < \varepsilon$ for each $y \in Y$. If x and y are points of a metric space, $d(x, y)$ denotes the distance from x to y. A map $r\colon X \to Y$ between compacta is *refinable* [**2**] if for each $\varepsilon > 0$ there is a surjective ε-mapping $f\colon X \to Y$ such that

$$d(r, f) = \sup\{d(r(x), f(x)) \mid x \in X\} < \varepsilon.$$

Such a map f is called an *ε-refinement* of r. A space X is *weakly infinite-dimensional* if for each countable family $\{(A_i, B_i) \mid i = 1, 2, 3, \ldots\}$ of pairs of disjoint closed sets in X there are partitions S_i between A_i and B_i with $\bigcap_{i=1}^{\infty} S_i = \varnothing$. A space X is *strongly infinite-dimensional* if X is not weakly infinite-dimensional. A space X is *countable-dimensional* if $X = \bigcup_{i=1}^{\infty} X_i$ with $\dim X_i \leq 0$ for each i. If X is countable-dimensional, then X is weakly infinite-dimensional (see [**3**, II2F p. 16]).

We need the following.

LEMMA [**5**, LEMMA 1]. *Let f be a map from a compactum X to an ANR A and $\varepsilon > 0$. Then there is a positive number $\delta > 0$ such that if g is any δ-mapping from X onto any compactum Y, then there is a map $h\colon Y \to A$ such that $d(f, hg) < \varepsilon$.*

By using the lemma, we show the following theorem.

THEOREM 1. *Every refinable map preserves weak infinite-dimension. In other words, there is no refinable map from a weakly infinite-dimensional compactum to a strongly infinite-dimensional compactum.*

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PROOF. Suppose that r is a refinable map from X onto Y and $\{(A_i, B_i) \mid i = 1, 2, \dots\}$ is a countable family of pairs of disjoint closed sets in Y. Since X is weakly infinite-dimensional and $\{(r^{-1}(A_i), r^{-1}(B_i)) \mid i = 1, 2, \dots\}$ is a countable family of pairs of disjoint closed sets in X, there exist separations S_i $(i = 1, 2, \dots)$ between $r^{-1}(A_i)$ and $r^{-1}(B_i)$ such that $\bigcap_{i=1}^{\infty} S_i = \varnothing$. Since X is compact, there is a natural number n such that $\bigcap_{i=1}^{n} S_i = \varnothing$. For each $i = 1, 2, \dots, n$, choose neighborhoods U_i of S_i in X such that $\bigcap_{i=1}^{n} U_i = \varnothing$. Let f_i: $X \to [0, 1]$ be a map $(i = 1, 2, \dots, n)$ such that

(1) $$f_i^{-1}(0) = r^{-1}(A_i), \qquad f_i^{-1}(1) = r^{-1}(B_i)$$

and

(2) $$f_i^{-1}(1/2) = S_i.$$

Also, for each $i = 1, 2, \dots, n$ choose neighborhoods V_i and W_i of $r^{-1}(A_i)$ and $r^{-1}(B_i)$, respectively, in X such that

(3) $$f_i(V_i) \subset [0, 1/8] \quad \text{and} \quad f_i(W_i) \subset [7/8, 1].$$

Since r is a refinable map and by using the lemma, we can easily see that there exists a sequence $\{r_j$: $X \to Y \mid j = 1, 2, \dots\}$ of maps and sequences $\{h_{ij}$: $Y \to [0, 1] \mid j = 1, 2, \dots\}$ of maps $(i = 1, 2, \dots, n)$ such that

(4) $$r_j \text{ is an } (1/j)\text{-refinement of } r$$

and

(5) $$d(f_i, h_{ij} r_j) < 1/j \quad \text{for each } i = 1, 2, \dots, n \text{ and each } j = 1, 2, \dots.$$

Now, we shall show that

(6) $$\limsup_j r_j^{-1}(A_i) \subset r^{-1}(A_i), \qquad i = 1, 2, \dots, n,$$

(7) $$\limsup_j r_j^{-1}(B_i) \subset r^{-1}(B_i), \qquad i = 1, 2, \dots, n,$$

and

(8) $$\limsup_j r_j^{-1} h_{ij}^{-1}(1/2) \subset f_i^{-1}(1/2) = S_i, \qquad i = 1, 2, \dots, n.$$

We will prove (6). Let $x_{jk} \in r_{jk}^{-1}(A_i)$ with $\lim_k x_{jk} = x$. Then by (4), we have

$$r(x) = \lim_k r(x_{jk}) = \lim r_{jk}(x_{jk}) \in A_i.$$

Hence $x \in r^{-1}(A_i)$, which implies (6). By (4) and (5), conditions (7) and (8) are similarly proved. By (6), (7) and (8), we can choose a sufficiently large number m such that

(9) $$1/m < 1/8,$$

(10) $$r_m^{-1}(A_i) \subset V_i \qquad (i = 1, 2, \dots, n),$$

(11) $$r_m^{-1}(B_i) \subset W_i \qquad (i = 1, 2, \dots, n)$$

and

(12) $$r_m^{-1} h_{im}^{-1}(1/2) \subset U_i \qquad (i = 1, 2, \dots, n).$$

Then we shall show that

(13) $$h_{im}(A_i) \subset [0, 1/4], \qquad i = 1, 2, \ldots, n,$$

and

(14) $$h_{im}(B_i) \subset [3/4, 1], \qquad i = 1, 2, \ldots, n.$$

Let $y \in A_i$ and choose $x \in r_m^{-1}(y) \subset r_m^{-1}(A_i)$. Note that $x \in V_i$ (see (10)). Then by (3), (5) and (9), we have

$$\begin{aligned} d(0, h_{im}(y)) &= d(0, h_{im} r_m(x)) \\ &\leqslant d(0, f_i(x)) + d(f_i(x), h_{im} r_m(x)) \\ &\leqslant 1/8 + 1/8 = 1/4. \end{aligned}$$

This implies (13). Condition (14) is similarly proved. Set $T_i = h_{im}^{-1}(1/2)$ for each $i = 1, 2, \ldots, n$. Then by (13) and (14), T_i is a separation between A_i and B_i ($i = 1, 2, \ldots, n$). Since each T_i maps under r_m^{-1} into U_i (see (12)) and the U_i do not intersect, neither do the T_i. This implies that Y is weakly infinite-dimensional. This completes the proof.

Next, we will show that refinable maps do not preserve strong infinite-dimension. More precisely, we give a refinable map $r\colon X \to Y$ such that X is a strongly infinite-dimensional AR and Y is a countable-dimensional AR. In particular, X and Y are quasi-homeomorphic. First, we will prove the following theorem.

THEOREM 2. *If Z is a compactum, then there is a sequence $C_1, C_2, \ldots$ of disjoint finite-dimensional compacta and a refinable map from the join $X = Z \vee C_1 \vee C_2 \vee \cdots$, onto the join $Y = C_1 \vee C_2 \vee \cdots$, where these joins are constructed so that* $\operatorname{diam}\{C_n\} \to 0$.

PROOF. Choose a point $* \in Z$. Let $((Z_n, z_n), f_n)$ be an inverse sequence of pointed compact polyhedra such that $\operatorname{inv}\lim((Z_n, z_n), f_n) = (Z, *)$ and let $p_n\colon Z \to Z_n$ be the projection maps. Set $C_n = p_n(Z)$ for each $n = 1, 2, \ldots$. Note that $\dim C_n < \infty$. By identifying the points $z_1, z_2, \ldots$, we obtain a compactum $(Y, *) = \bigvee_{n=1}^{\infty}(C_n, z_n)$ such that $\operatorname{diam}\{C_n\} \to 0$. By identifying the points $* \in Z$ and $* \in Y$, we obtain a compactum $(X, *) = (Z, *) \vee (Y, *)$. Now, let us define a map $r\colon X \to Y$ by

$$r(x) = \begin{cases} *, & \text{if } x \in Z, \\ x, & \text{if } x \in Y. \end{cases}$$

Then we shall show that $r\colon X \to Y$ is a refinable map (cf. [**4**, Example 2.6]). In fact, for a given $\varepsilon > 0$ choose a natural number m such that

(1) $\operatorname{diam} C_m < \varepsilon/2$, and

(2) the projection map $p_m\colon Z \twoheadrightarrow C_m \subset Z_m$ is an $\varepsilon/2$-mapping.

Define a map $g\colon X \to Y$ by

$$g(x) = \begin{cases} p_m(x), & \text{if } x \in Z, \\ *, & \text{if } x \in C_m, \\ x, & \text{otherwise.} \end{cases}$$

Then $d(r, g) < \varepsilon$ and diam $g^{-1}(y) < \varepsilon$ for each $y \in Y$. Hence r is a refinable map. This completes the proof.

COROLLARY 1. *If $Z = Q$ is the Hilbert cube, then $X = Q \vee C_1 \vee C_2 \vee \cdots$ is a strongly infinite-dimensional AR and its refinable image $Y = C_1 \vee C_2 \vee \cdots$ is a countable-dimensional AR.*

PROOF. Let (I^n, f_n) be the inverse sequence such that $I^n = [0, 1]^n$ and $f_n(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n)$ for $(x_1, x_2, \ldots, x_{n+1}) \in I^{n+1}$. Then $Q = \operatorname{inv\,lim}(I^n, f_n)$. By the proof of Theorem 2, there is a refinable map from $Q \vee I^1 \vee I^2 \vee \cdots$ onto $I^1 \vee I^2 \vee \cdots$. Then $Q \vee I^1 \vee I^2 \vee \cdots$ is a strongly infinite-dimensional AR and $I^1 \vee I^2 \vee \cdots$ is a countable-dimensional AR.

In [7], R. Pol showed that there exists a weakly infinite-dimensional compactum which is not countable-dimensional. By using Pol's example, we have the following

COROLLARY 2. *If Z is Pol's example, then $Z \vee C_1 \vee C_2 \vee \cdots$ is a weakly infinite-dimensional compactum not of countable-dimension and its refinable image is a countable-dimensional compactum.*

Finally, the following problem is raised: Does each refinable map preserve countable-dimension?

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ADDED IN PROOF. There is a compact AR which is weakly infinite-dimensional but not countable dimensional. In fact, let Z be Pol's example. Choose an inverse sequence $\underline{Z} = \{Z_i, p_{ii+1}\}$ of compact polyhedra such that Invlim $\underline{Z} = Z$. Note that the countable sum of closed subsets, which are weakly infinite-dimensional, is also weakly infinite-dimensional. Then the space of the inverse sequence $\underline{Z}$ ($= S\underline{Z}$) is a compact AR (see [J. Krasinkiewicz, *On a method of constructing ANR-sets. An application of inverse limits*, Fund. Math. **92** (1976), 100, Definition 3]) which is weakly infinite-dimensional but not countable dimensional. Hence, Theorem 2 implies that there are compact AR's X, Y and a refinable map $r: X \to Y$ such that Y is countable dimensional and X is weakly infinite-dimensional but not countable dimensional. Note that X and Y are quasi-homeomorphic (cf. Corollary 2).

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