A NOTE ON INFINITE-DIMENSION UNDER REFINABLE MAPS

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Abstract. It is shown that refinable maps preserve weak infinite-dimension, but not strong infinite-dimension.

The purpose of this note is to show that refinable maps preserve weak infinite-dimension, but not strong infinite-dimension. Under a refinable map between compacta, the domain and image must have the same finite-dimension or must both have infinite-dimension (see [4, Theorem 1.8(4); 6, Theorem I, 16]).

The term compactum is used to mean a compact metric space. A map \( f : X \to Y \) between compacta is said to be an \( \varepsilon \)-mapping, \( \varepsilon > 0 \), if \( \text{diam} \ f^{-1}(y) < \varepsilon \) for each \( y \in Y \). If \( x \) and \( y \) are points of a metric space, \( d(x, y) \) denotes the distance from \( x \) to \( y \). A map \( r : X \to Y \) between compacta is refinable [2] if for each \( \varepsilon > 0 \) there is a surjective \( \varepsilon \)-mapping \( f : X \to Y \) such that

\[
\text{d}(r, f) = \sup \{ d(r(x), f(x)) \mid x \in X \} < \varepsilon.
\]

Such a map \( f \) is called an \( \varepsilon \)-refinement of \( r \). A space \( X \) is weakly infinite-dimensional if for each countable family \( \{(A_i, B_i) \mid i = 1, 2, 3, \ldots \} \) of pairs of disjoint closed sets in \( X \) there are partitions \( S_i \) between \( A_i \) and \( B_i \) with \( \bigcap_{i=1}^{\infty} S_i = \emptyset \). A space \( X \) is strongly infinite-dimensional if \( X \) is not weakly infinite-dimensional. A space \( X \) is countable-dimensional if \( X = \bigcup_{i=1}^{\infty} X_i \) with \( \dim X_i \leq 0 \) for each \( i \). If \( X \) is countable-dimensional, then \( X \) is weakly infinite-dimensional (see [3, I12F p. 16]).

We need the following.

Lemma [5, Lemma 1]. Let \( f \) be a map from a compactum \( X \) to an ANR \( A \) and \( \varepsilon > 0 \). Then there is a positive number \( \delta > 0 \) such that if \( g \) is any \( \delta \)-mapping from \( X \) onto any compactum \( Y \), then there is a map \( h : Y \to A \) such that \( d(f, hg) < \varepsilon \).

By using the lemma, we show the following theorem.

Theorem 1. Every refinable map preserves weak infinite-dimension. In other words, there is no refinable map from a weakly infinite-dimensional compactum to a strongly infinite-dimensional compactum.
Proof. Suppose that \( r \) is a refinable map from \( X \) onto \( Y \) and \( \{ (A_i, B_i) \mid i = 1, 2, \ldots \} \) is a countable family of pairs of disjoint closed sets in \( Y \). Since \( X \) is weakly infinite-dimensional and \( \{ (r^{-1}(A_i), r^{-1}(B_i)) \mid i = 1, 2, \ldots \} \) is a countable family of pairs of disjoint closed sets in \( X \), there exist separations \( S_i (i = 1, 2, \ldots) \) between \( r^{-1}(A_i) \) and \( r^{-1}(B_i) \) such that \( \bigcap_{i=1}^{\infty} S_i = \emptyset \). Since \( X \) is compact, there is a natural number \( n \) such that \( \bigcap_{i=1}^{n} U_i = \emptyset \). For each \( i = 1, 2, \ldots, n \), choose neighborhoods \( U_i \) of \( S_i \) in \( X \) such that \( \bigcap_{i=1}^{n} U_i = \emptyset \). Let \( f_i: X \to [0,1] \) be a map \((i = 1, 2, \ldots, n)\) such that

\[
(1) \quad f_i^{-1}(0) = r^{-1}(A_i), \quad f_i^{-1}(1) = r^{-1}(B_i)
\]

and

\[
(2) \quad f_i^{-1}(1/2) = S_i.
\]

Also, for each \( i = 1, 2, \ldots, n \) choose neighborhoods \( V_i \) and \( W_i \) of \( r^{-1}(A_i) \) and \( r^{-1}(B_i) \), respectively, in \( X \) such that

\[
(3) \quad f_i(V_i) \subset [0, 1/8] \quad \text{and} \quad f_i(W_i) \subset [7/8, 1].
\]

Since \( r \) is a refinable map and by using the lemma, we can easily see that there exists a sequence \( \{r_j: X \to Y \mid j = 1, 2, \ldots \} \) of maps and sequences \( \{h_{ij}: Y \to [0,1] \mid i = 1, 2, \ldots, n \} \) of maps \((i = 1, 2, \ldots, n)\) such that

\[
(4) \quad r_j \text{ is an } (1/j)\text{-refinement of } r
\]

and

\[
(5) \quad d(f_i, h_{ij}) < 1/j \quad \text{for each } i = 1, 2, \ldots, n \text{ and each } j = 1, 2, \ldots.
\]

Now, we shall show that

\[
(6) \quad \limsup_j r_j^{-1}(A_i) \subset r^{-1}(A_i), \quad i = 1, 2, \ldots, n,
\]

\[
(7) \quad \limsup_j r_j^{-1}(B_i) \subset r^{-1}(B_i), \quad i = 1, 2, \ldots, n,
\]

and

\[
(8) \quad \limsup_j r_j^{-1} h_{ij}^{-1}(1/2) \subset f_i^{-1}(1/2) = S_i, \quad i = 1, 2, \ldots, n.
\]

We will prove (6). Let \( x_jk \in r^{-1}_j(A_i) \) with \( \lim_k x_jk = x \). Then by (4), we have

\[
r(x) = \lim_k r(x_jk) = \lim_j r_jk(x_jk) \in A_i.
\]

Hence \( x \in r^{-1}(A_i) \), which implies (6). By (4) and (5), conditions (7) and (8) are similarly proved. By (6), (7) and (8), we can choose a sufficiently large number \( m \) such that

\[
(9) \quad 1/m < 1/8,
\]

\[
(10) \quad r_m^{-1}(A_i) \subset V_i \quad (i = 1, 2, \ldots, n),
\]

\[
(11) \quad r_m^{-1}(B_i) \subset W_i \quad (i = 1, 2, \ldots, n)
\]

and

\[
(12) \quad r_m^{-1} h_{im}^{-1}(1/2) \subset U_i \quad (i = 1, 2, \ldots, n).
\]
Then we shall show that
\begin{equation}
\tag{13}
 h_i \circ (A_i) \subset [0, 1/4], \quad i = 1, 2, \ldots, n,
\end{equation}
and
\begin{equation}
\tag{14}
 h_i \circ (B_i) \subset [3/4, 1], \quad i = 1, 2, \ldots, n.
\end{equation}
Let \( y \in A_i \) and choose \( x \in r_m^{-1}(y) \subset r_m^{-1}(A_i) \). Note that \( x \in V_i \) (see (10)). Then by (3), (5) and (9), we have
\[
d(0, h_i(x)) = d(0, h_i (r_m(x)) \\
\leq d(0, f_i(x)) + d(f_i(x), h_i (r_m(x))) \\
\leq 1/8 + 1/8 = 1/4.
\]
This implies (13). Condition (14) is similarly proved. Set \( T_i = h_i^{-1}(1/2) \) for each \( i = 1, 2, \ldots, n \). Then by (13) and (14), \( T_i \) is a separation between \( A_i \) and \( B_i \) (\( i = 1, 2, \ldots, n \)). Since each \( T_i \) maps under \( r_m^{-1} \) into \( U_i \) (see (12)) and the \( U_i \) do not intersect, neither do the \( T_i \). This implies that \( Y \) is weakly infinite-dimensional. This completes the proof.

Next, we will show that refinable maps do not preserve strong infinite-dimension. More precisely, we give a refinable map \( r: X \to Y \) such that \( X \) is a strongly infinite-dimensional AR and \( Y \) is a countable-dimensional AR. In particular, \( X \) and \( Y \) are quasi-homeomorphic. First, we will prove the following theorem.

**Theorem 2.** If \( Z \) is a compactum, then there is a sequence \( C_1, C_2, \ldots \) of disjoint finite-dimensional compacta and a refinable map from the join \( X = Z \vee C_1 \vee C_2 \vee \cdots \), onto the join \( Y = C_1 \vee C_2 \vee \cdots \), where these joins are constructed so that \( \text{diam}(C_n) \to 0 \).

**Proof.** Choose a point \( \ast \in Z \). Let \( ((Z_n, z_n), f_n) \) be an inverse sequence of pointed compact polyhedra such that \( \text{invlim}((Z_n, z_n), f_n) = (Z, \ast) \) and let \( p_n: Z \to Z_n \) be the projection maps. Set \( C_n = p_n(Z) \) for each \( n = 1, 2, \ldots \). Note that \( \dim C_n < \infty \).

By identifying the points \( z_1, z_2, \ldots \), we obtain a compactum \( (Y, \ast) = \bigvee_{n=1}^{\infty} (C_n, z_n) \) such that \( \text{diam}(C_n) \to 0 \). By identifying the points \( \ast \in Z \) and \( \ast \in Y \), we obtain a compactum \( (X, \ast) = (Z, \ast) \vee (Y, \ast) \). Now, let us define a map \( r: X \to Y \) by
\[
r(x) = \begin{cases} 
\ast, & \text{if } x \in Z, \\
x, & \text{if } x \in Y.
\end{cases}
\]
Then we shall show that \( r: X \to Y \) is a refinable map (cf. [4, Example 2.6]). In fact, for a given \( \epsilon > 0 \) choose a natural number \( m \) such that
\begin{enumerate}
\item \( \text{diam} C_m < \epsilon/2 \), and
\item the projection map \( p_m: Z \to C_m \subset Z_m \) is an \( \epsilon/2 \)-mapping.
\end{enumerate}
Define a map \( g: X \to Y \) by
\[
g(x) = \begin{cases} 
p_m(x), & \text{if } x \in Z, \\
\ast, & \text{if } x \in C_m, \\
x, & \text{otherwise}.
\end{cases}
\]
Then $d(r, g) < \varepsilon$ and $\text{diam } g^{-1}(y) < \varepsilon$ for each $y \in Y$. Hence $r$ is a refinable map. This completes the proof.

**Corollary 1.** If $Z = Q$ is the Hilbert cube, then $X = Q \vee C_1 \vee C_2 \vee \cdots$ is a strongly infinite-dimensional $AR$ and its refinable image $Y = C_1 \vee C_2 \vee \cdots$ is a countable-dimensional $AR$.

**Proof.** Let $(I^n, f_n)$ be the inverse sequence such that $I^n = [0, 1]^n$ and $f_n(x_1, x_2, \ldots, x_n, x_{n+1}) = (x_1, x_2, \ldots, x_n)$ for $(x_1, x_2, \ldots, x_{n+1}) \in I^{n+1}$. Then $Q = \text{inv lim}(I^n, f_n)$. By the proof of Theorem 2, there is a refinable map from $Q \vee I^1 \vee I^2 \vee \cdots$ onto $I^1 \vee I^2 \vee \cdots$. Then $Q \vee I^1 \vee I^2 \vee \cdots$ is a strongly infinite-dimensional $AR$ and $I^1 \vee I^2 \vee \cdots$ is a countable-dimensional $AR$.

In [7], R. Pol showed that there exists a weakly infinite-dimensional compactum which is not countable-dimensional. By using Pol's example, we have the following

**Corollary 2.** If $Z$ is Pol's example, then $Z \vee C_1 \vee C_2 \vee \cdots$ is a weakly infinite-dimensional compactum not of countable-dimension and its refinable image is a countable-dimensional compactum.

Finally, the following problem is raised: Does each refinable map preserve countable-dimension?

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**ADDED IN PROOF.** There is a compact $AR$ which is weakly infinite-dimensional but not countable dimensional. In fact, let $Z$ be Pol's example. Choose an inverse sequence $Z = \{Z_i, p_{ii+1}\}$ of compact polyhedra such that $\text{inv lim } Z = Z$. Note that the countable sum of closed subsets, which are weakly infinite-dimensional, is also weakly infinite-dimensional. Then the space of the inverse sequence $Z (= SZ)$ is a compact AR (see [J. Krasinkiewicz, *On a method of constructing ANR-sets. An application of inverse limits*, Fund. Math. 92 (1976), 100, Definition 3]) which is weakly infinite-dimensional but not countable dimensional. Hence, Theorem 2 implies that there are compact AR's $X$, $Y$ and a refinable map $r: X \to Y$ such that $Y$ is countable dimensional and $X$ is weakly infinite-dimensional but not countable dimensional. Note that $X$ and $Y$ are quasi-homeomorphic (cf. Corollary 2).

**References**


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