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RATIONAL AUTOMORPHISMS OF GRASSMANN MANIFOLDS

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ABSTRACT. The homotopy class of a self map of a complex projective space is well known to be classified by a degree detected in two dimensional cohomology. An analogous result is proved for the *rationalization* of the Grassmann manifold of complex n-planes in complex N-space, provided $N \neq 2n$ and the degree is not zero.

Let $G_n(\mathbf{C}^{n+k})$ denote the complex Grassmann manifold $U(n+k)/(U(n) \times U(k))$, and let R be a subring of the rational numbers $\mathbf{Q}$ containing the integer $\mathbf{Z}$. By the grading endomorphism φ of degree $\lambda \in R$ we mean the endomorphism defined by $\varphi(x) = \lambda^m x$ for $x \in H^{2m}(G_n(\mathbf{C}^{n+k}); R)$. It is an automorphism if λ is invertible in R.

THEOREM. *If φ is an endomorphism of the graded ring $H^*(G_n(\mathbf{C}^{n+k}); R)$, $n \neq k$, and φ is not identically zero on $H^2(G_n(\mathbf{C}^{n+k}); R)$, then φ is a grading endomorphism.*

REMARKS. 1. Since $H^2(G_n(\mathbf{C}^{n+k}); R) \approx R$, the degree λ of φ is of course determined by $\varphi(x) = \lambda x$ for any choice of $x \neq 0$ in $H^2(G_n(\mathbf{C}^{n+k}); R)$ (for example, $x =$ the first Chern class of the canonical $\mathbf{C}^n$-bundle).

2. If φ is zero on $H^2(G_n(\mathbf{C}^{n+k}); R)$, we conjecture that φ is in fact the zero endomorphism. This is proved in **[GH1]** when $k \geqslant 2n^2 - n - 1$ and has been checked for a few other cases involving small values of n and k.

3. When $n = k$, swapping the first n coordinates of $\mathbf{C}^{2n}$ with the last n coordinates induces an involution f of $G_n(\mathbf{C}^{2n})$. Any endomorphism which is nonzero on $H^2(G_n(\mathbf{C}^{2n}))$ is either a grading endomorphism or the composition of a grading endomorphism with f^*. (Note that $-f^* = \mathrm{id}$ if $n = k = 1$.) A proof may be found in **[Hf]**, where the approach of **[Br]** is combined with some additional structure of the ring $H^*(G_n(\mathbf{C}^{n+k}); R)$. See Remark 9 for more details.

4. The previous remarks and our theorem may be restated as follows. Every endomorphism which is not zero on $H^2(G_n(\mathbf{C}^{n+k}); R)$ is the composition of a grading endomorphism with an endomorphism induced from an element of the Weyl group of $U(n) \times U(k) \subset U(n+k)$. The conjecture in Remark 2 is that one can omit the phrase "on $H^2(G_n(\mathbf{C}^{n+k}); R)$." The analogous statement for more general homogeneous spaces of maximal rank is not true, since there is (at least) one additional type of endomorphism (cf. **[GH2]**).

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From [**GH2**] we know that endomorphisms of the rational cohomology ring of $G_n(\mathbf{C}^{n+k})$ are in one-to-one correspondence with homotopy classes of self maps of the rationalization of $G_n(\mathbf{C}^{n+k})$. Hence our present theorem has the following geometric corollary.

COROLLARY. *When $n \neq k$, the homotopy classes of self maps of the rationalization of $G_n(\mathbf{C}^{n+k})$ which are nonzero on two dimensional cohomology form a group under composition which is isomorphic to the nonzero rational numbers under multiplication.*

REMARKS. 5. The corollary says in other words that the group of rational automorphisms of $G_n(\mathbf{C}^{n+k})$ is as small as possible. The manifold thus has a kind of "rigidity" which not surprisingly shows up in other ways as well. For applications of our theorem to group actions, see [**Ba** and **L**]; to fixed points and coincidences of maps. [**GH**1 and **Hf**]; and to triviality of the genus set, [**GM**].

6. As noted above, if $k \geqslant 2n^2 - n - 1$ we can omit the phrase "on two dimensional cohomology".

7. For a degree to be realized by a self map of $G_n(\mathbf{C}^{n+k})$ itself (rather than of the rationalization), it must of course be an integer. One knows from general principles (cf. [**GH2**]) that infinitely many integers are so realized, but it is a difficult obstruction theory problem to determine precisely which integers these are (cf. [**F**] and, for some related results, [**H**]).

PROOF OF THE THEOREM. Let c_i denote the ith Chern class of the canonical $\mathbf{C}^n$-bundle over $G_n(\mathbf{C}^{n+k})$. If φ is an endomorphism of $H^*(G_n(\mathbf{C}^{n+k}); R)$, define a_i to be the coefficient of c_i in $\varphi(c_i)$. Since the ring is free through dimension $2k$ and we may assume, without loss of generality, that $k \geqslant n$, the a_i's are well defined. The main result of [**Br**] states that if all $a_i \neq 0$, then φ is the grading endomorphism of degree a_1. Although the proof is written for $R = \mathbf{Z}$ it works as well for any larger subring of $\mathbf{Q}$. It now remains to show that $a_1 \neq 0$ implies that $a_i \neq 0$ for all $1 \leqslant i \leqslant n$.

Since $G_n(\mathbf{C}^{n+k})$ is a Kähler manifold, the 2-form representing the Kähler metric has maximal cup length. It follows that $c_1^{nk} = Mx$, where x is dual to the fundamental homology class and $M \neq 0$ is an integer. Applying φ yields $0 \neq a_1^{nk} c_1^{nk} = M\varphi(x)$, and so $\varphi(x) \neq 0$. Given any nonzero class $y \in H^{2m}(G_n(\mathbf{C}^{n+k}); R)$, by Poincaré duality there is a $z \in H^{2nk-2m}(G_n(\mathbf{C}^{n+k}); R)$ such that $y \cup z$ is a nonzero multiple of x. Hence $0 \neq \varphi(y \cup z) = \varphi(y) \cup \varphi(z)$, and so $\varphi(y) \neq 0$. Thus φ is injective, so $\varphi \otimes \mathbf{Q}$ is an isomorphism and induces an isomorphism on the quotient algebra of irreducibles. Since c_i represents the single class of irreducibles in dimension $2i$, it follows as desired that $a_i \neq 0$ for $1 \leqslant i \leqslant n$.

REMARKS. 8. The proof of the main result of [**Br**] uses the following approach. The cohomology ring $H^*(G_n(\mathbf{C}^{n+k}); R)$ is generated by the Chern classes $c_1, \ldots, c_n$ and its ideal of relators is generated by elements $R_{1+k}, \ldots, R_{n+k}$ with R_i in dimension $2i$ (cf. [**GH1**]). Since φ is assumed to preserve dimension, we may write $\varphi(c_i) = \Sigma a_x x$ where x runs through all monomials in the c_i's of dimension $2i$ and $a_x \in R$. The requirement that $\varphi(R_{1+k})$ must be a multiple of R_{1+k} restricts the possibilities for the coefficients a_x. In [**GH1**] it was shown that when $k \geqslant 2n^2 - n - 1$, this restriction already requires φ to be a grading endomorphism. To treat the remaining cases in **[Br]**, it was necessary to consider the restrictions on the a_x resulting from the additional requirements that $\varphi(R_i)$ should be in the ideal generated by $R_{1+k}, \ldots, R_i$, where $1 + k < i \leqslant n + k$. The restrictions which come from these higher dimensions are less convenient than those used in **[GH1]**, and the proof is long and very computational. The coefficients a_x are, for the most part, considered in a lexicographical order on their subscripts.

9. The innovation in **[Hf]** is first to show, via the "hard Lefschetz theorem," that $a_i = \varepsilon_i a_1^i$, where $\varepsilon_i \in \{1, -1\}$ and the coefficient of c_i in $\varphi(c_i)$ is denoted by a_i (rather than by a_{c_i} as in Remark 8). Given this preliminary information, a substantially shortened version of the argument given in **[Br]** suffices to complete the proof.

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