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A TRANSCENDENCE MEASURE FOR SOME SPECIAL VALUES OF ELLIPTIC FUNCTIONS

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ABSTRACT. Among T. Schneider's results is the following: Let $\wp(z)$ be the Weierstrass elliptic function with algebraic invariants. If $\wp(u)$ and β are both algebraic, $\beta \notin K_\tau$, then $\wp(\beta u)$ is transcendental. In this paper we provide a transcendence measure for this value.

Let $P(X)$ be a nonzero polynomial, with integral coefficients, of degree d and height h, and put $t = d + \log h$. Then there is an effectively computable constant C, which does not depend on $P(X)$, such that:

(A) If $\wp(z)$ has complex multiplication then $\log |P(\wp(\beta u))| > -Cd^2t^2(\log t)^4$.

(B) If $\wp(z)$ does not have complex multiplication then $\log |P(\wp(\beta u))| > -Cd^6t^2(\log t)^{14}$.

Introduction. The aim of this paper is to provide transcendence measures for the Weierstrass $\wp$-function evaluated at numbers from a particular class of values. We derive our measures under the assumption that the invariants for $\wp(z)$ are algebraic integers. In particular, we prove the following result.

THEOREM. *Suppose that u is a nontorsion algebraic point of $\wp(z)$ and β is an algebraic number, $\beta \notin K_\tau$. Let $P(X)$ be a nonzero integral polynomial with $d = \deg P$, $h = \operatorname{ht} P$, and $t = d + \log h$. Then there is an effectively computable constant C (depending only on u, β, $\wp$) such that*:

(A) *If $\wp(z)$ has complex multiplication then*

$$\log |P(\wp(\beta u))| > -Cd^2t^2(\log t)^4.$$

(B) *If $\wp(z)$ does not have complex multiplication then*

$$\log |P(\wp(\beta u))| > -Cd^6t^2(\log t)^{14}.$$

Preliminaries. We take ω_1, ω_2 for the generators of the period lattice Ω of $\wp(z)$, that is, $\Omega = Z\omega_1 + Z\omega_2$; $\wp$ is periodic with respect to Ω and is meromorphic with poles on Ω. If we take $g_2 = \sum_{\omega\in\Omega^*}\omega^{-4}$ and $g_3 = \sum_{\omega\in\Omega^*}\omega^{-6}$, where $\Omega^* = \Omega - \{0\}$, then $\wp$ satisfies the differential equation $(\wp'(z))^2 = 4\wp^3(z) - g_2\wp(z) - g_3$. The numbers g_2 and g_3 are called the invariants of $\wp$. Throughout this paper we assume that $g_2/4$ and $g_3/4$ are algebraic integers.

We recall that $\wp$ is said to have complex multiplication when $\tau = \omega_1/\omega_2$ is algebraic, in which case τ is a quadratic irrationality. In this situation the mapping

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$\sigma: \wp(z) \to \wp(\sigma z)$ is an endomorphism of $\mathbf{Q}/\Omega$ for all $\sigma \in \mathcal{O}$. $\mathcal{O} = Z + Z\tau$ is the set of multiplications of $\wp$. We also use the notation

$$\mathcal{O}(S) = \{\sigma \in \mathcal{O} \mid \sigma = s_1 + s_2\tau \text{ with } |s_i| \leq S \text{ for } i = 1,2\}.$$

When τ is not algebraic then $\mathcal{O} = Z$ and $\mathcal{O}(S)$ consists of integers σ with $|\sigma| \leq S$. In the case of complex multiplication we put $K_\tau = \mathbf{Q}(\tau)$, otherwise $K_\tau = \mathbf{Q}$.

It is a result of T. Schneider that for $\beta \notin K_\tau$ the functions $\wp(z)$ and $\wp^*(z) = \wp(\beta z)$ satisfy not all of g_2, g_3, g_2^*, g_3^*, β, $\wp(u)$, and $\wp^*(u)$ are algebraic, where we have let g_2^* and g_3^* denote the invariants of $\wp^*(z)$. From our hypothesis that β and $\wp(u)$ are algebraic, and $\beta \notin K_\tau$, it follows that $\wp(\beta u)$ is transcendental.

Our method of proof, due to A. O. Gelfond, will require that we define parameters D, K, and S and then construct a polynomial $P^*(X, Y)$ of total degree at most D such that the auxiliary function $\Phi(z) = P^*(\wp(z), \wp(\beta z))$ satisfies $\Phi^{(k)}(\sigma u) = 0$ for $0 \leq k \leq K$ and $\sigma \in \mathcal{O}(S)$. Our construction of the polynomial $P^*(X, Y)$ requires the following lemma.

LEMMA 1. *Let S and R be rational integers and F a number field. Consider the system of equations $\sum_{i=1}^{S} a_{ij} z_i = 0$ for $1 \leq j \leq R$, where a_{ij} satisfy* (i) $a_{ij} \in F[X_1, \dots, X_6]$ *with coefficients which are integers in F with sizes bounded by A, and* (ii) $\deg_{X_k} a_{ij} \leq d_k$ $(1 \leq k \leq 6)$ *with* $1 = d_4 = d_5 = d_6$. *Then, provided that $S \leq [F:\mathbf{Q}]2^6 R$, the system has a nonzero solution $(z_1, \dots, z_S)$ with $z_i \in Z[X_1, X_2, X_3]$ with* $\deg_{X_k} z_i \leq d_k$ $(k = 1,2,3)$ *and*

$$\operatorname{ht}(z_i) \leq \left(AS \prod_{k=1}^{3} (1 + d_k)^2 \right)^{[F:\mathbf{Q}]2^6 R/(S - [F:\mathbf{Q}]2^6 R)}.$$

PROOF. Substitute z_i into each equation, leaving the coefficients of z_i undetermined. If we then multiply out these expressions and collect together the coefficients of each monomial $X_4^{e(4)} X_5^{e(5)} X_6^{e(6)}$, $e(k) \in \{0, 1\}$, we obtain eight polynomials $P_{e(4),e(5),e(6)}(X_1, X_2, X_3)$. Each of these polynomials contain $\prod_{k=1}^{3}(1 + 2d_k)$ monomials with coefficients in terms of $S\prod_{k=1}^{3}(1 + d_k)$ unknowns (these are the "undetermined coefficients" of the z_i). Setting the coefficient of each of the monomials equal to zero yields $8\prod_{k=1}^{3}(1 + 2d_k)$ equations in $S\prod_{k=1}^{3}(1 + d_k)$ unknowns. These equations have coefficients which are algebraic integers of size bounded by $A\prod_{k=1}^{3}(1 + d_k)$. Therefore by the standard Thue-Siegel lemma [**5**, Lemma 1.3.2] we can solve over the rational integers provided that

$$S \prod_{k=1}^{3} (1 + d_k) \leq [F:\mathbf{Q}] 8 \prod_{k=1}^{3} (1 + 2d_k).$$

It suffices that $S \leq [F:\mathbf{Q}]2^6 R$. Then the absolute value of the solutions, which are the coefficients of the polynomials z_i, are majorized by the bound given above.

In the next lemma we take $\sigma(z)$ to be the Weierstrass sigma function. We recall that $\sigma(z)$ is defined with respect to the lattice Ω so that $\sigma(z)$ is entire with zeros of order 1 at each lattice point. It is then basic that $\sigma^2(z)\wp(z)$ and $\sigma^3(z)\wp'(z)$ are entire functions.

LEMMA 2. *Let u be a nontorsion point for $\wp(z)$ and $\alpha_1,\ldots,\alpha_n$ K_τ-linearly independent numbers. There exists a constant C_0 (depending only on $u, \alpha_1,\ldots,\alpha_n$) such that the set*

$$\mathcal{O}'(s) = \left\{ s \in \mathcal{O}(S) \mid |\sigma(s\alpha_i u)| \geq C_0 e^{-S^2} \text{ for all } i = 1,\ldots,n \right\}$$

satisfies card $\mathcal{O}'(S) \geq (n+1)^{-1}$card $\mathcal{O}(S)$.

PROOF. We consider separately the cases when $\wp$ does or does not have complex multiplication.

If $\wp(z)$ does not have complex multiplication we let $t = n + 2$ and define a constant c' by

$$c' = \min\{ |s\alpha_i u - \omega| \mid 1 \leq i \leq n, \omega \in \Omega, s \in \mathcal{O}(2t) \}.$$

For each $p \in N$ we let $R_p = \{ r \mid pt \leq r < (p+1)t \}$. Then $\mathcal{O}(S)$ is contained in the disjoint union of the sets R_p for $0 \leq p \leq Sn^{-1}$. We begin by showing that each set R_p contains at most n multiplications s such that $|s\alpha_i u - \omega| < \frac{1}{2}c'$ for some $1 \leq i \leq n$ and some $\omega \in \Omega$. This follows since if there are at least $n + 1$ such multiplications, then two of them, say s_1 and s_2, correspond to the same α_i. In this case $s_0 = s_1 - s_2$ lies in $\mathcal{O}(2t)$ and contradicts our choice of c'.

Therefore at least one of the multiplications in each R_p satisfies $|s\alpha_i u - \omega| > \frac{1}{2}c'$ for all i and for all $\omega \in \Omega$. Yet $p \leq (n+1)^{-1}S$ implies that $R_p \subseteq \mathcal{O}(S)$ and therefore $\mathcal{O}(S)$ contains at least $(n+1)^{-1}S$ of these multiplications.

In this situation where $\wp$ has complex multiplication we take $t = [\sqrt{n+1}]$ and define c'' as we defined c' above. For each pair of integers p, q we let $R_{p,q} = \{ r + s\tau \mid pt \leq r < (p+1)t,\ qt \leq s < (q+1)t \}$. Arguing as above we see that at least $(n+1)^{-1}S^2$ of the multiplications s in $\mathcal{O}(S)$ satisfy $|s\alpha_i u - \omega| > \frac{1}{2}c''$ for all i and all $\omega \in \Omega$.

In either case let $C_0 = \min\{c', c''\}$ and apply Lemma 7.1 of [4] to obtain the desired result.

Once we have constructed our auxiliary function with zeros of order K at the points su for $s \in \mathcal{O}'(S)$, we will need an estimate for the total order of zeros at these points. This estimate is provided by the following result of Brownawell and Masser.

LEMMA 3. *For a nonnegative integer N let $\delta_0,\ldots,\delta_N$ be complex numbers and let $z_0,\ldots,z_N$ be points which are mutually incongruent modulo Ω. Suppose that $\wp(z)$ and $\wp(az + \delta_i)$ are analytic at $z = z_i$ for some nonzero a and for each i, $0 \leq i \leq N$. Then for $P(X, Y) \in C[X, Y]$, a nonzero polynomial with $\deg_X \leq L$ and $\deg_Y \leq M$, such that the functions $\Theta_i(z) = P(\wp(z), \wp(az + \delta_i))$ $(0 \leq i \leq N)$ are not identically zero, we have the estimate*

$$\sum_{i=0}^{N} \operatorname{ord}_{z=z_i} \Theta_i(z) \leq 8(L+1)(M+1) + 2MN.$$

PROOF. See Brownawell and Masser [1, Theorem 1].

We will also need to bound the modulus of these nonzero values of our auxiliary function. This we achieve through an application of Lemma 2 and the following lemma.

LEMMA 4 (SCHWARZ'S LEMMA). *Let $R > 2r$ be positive numbers and f a nonzero function of one variable which is continuous in the disc $|z| \leqslant R$ and analytic inside. We let $V_f(0, r)$ denote the number of zeros (counting multiplicities) of f in the disc $|z| < r$. Then*

$$\log |f|_r \leqslant \log |f|_R - V_f(0, r)\log(R/2r),$$

where $|f|_r = \sup_{z=r} |f(z)|$.

PROOF. See Waldschmidt [**5**, Lemma 1.3.1, p. 1.10].

Finally, for any number field F we define the F-height of a polynomial $P(X) = \Sigma_{i=0}^{d} a_i X^i$ by

$$h_F(P) = \sum_\nu N_\nu \log\max(1, |a_0|_\nu, \ldots, |a_d|_\nu),$$

where the sum is over all normalized valuations ν of F over $\mathbf{Q}$, and N_ν denotes the local degree of ν. For a nonzero element α of F we put $h_F(\alpha) = \Sigma_\nu N_\nu \log\max(1, |\alpha|_\nu)$. We then have the fundamental inequality:

LEMMA 5. *For $\alpha \in F^*$, $-h_F(\alpha) < \log|\alpha|$.*

PROOF. By the product formula, $1 = \prod_\nu |\alpha|_\nu^{N_\nu}$, we have

$$0 = \sum_\nu N_\nu \log|\alpha|_\nu < \log|\alpha| + \sum_\nu N_\nu \log\max(1, |\alpha|_\nu) = \log|\alpha| + h_F(\alpha).$$

Hence $-h_F(\alpha) < \log|\alpha|$.

This lemma will be used to produce the desired lower bound for $|P(\wp(\beta u))|$.

In all that follows we take $F = \mathbf{Q}(g_2, g_3, \wp(u), \wp'(u))$.

PROOF (THEOREM A). Let $P(X)$ be an integral polynomial with $d = \deg P$ and $h_F = h_F(P)$ satisfying $d \geqslant 1$, $h_F \geqslant 0$. Assume for the moment that $P(X)$ is monic and irreducible over F. The general case will follow easily from this. Notice that if $P(X) = X$ then it is obvious that there exists a constant C as above satisfying $\log|\wp(\beta u)| > -C$.

Take C_1 to be a sufficiently large constant and define parameters

$$D = \left[C_1^9\, dt(1 + \log t)^2\right], \qquad K = \left[C_1^{13}\, dt^2(1 + \log t)^3\right], \tag{1}$$
$$S = \left[C_1^2 d^{1/2}(1 + \log t)^{1/2}\right]$$

where $t = d + h_F$.

We will use an auxiliary function of the form

$$\Phi(z) = \sum_{l=0}^{D} \sum_{m=0}^{D} P_{lm}(\wp(\beta u))(\wp(z))^l(\wp(\beta z))^m \tag{2}$$

with the polynomials P_{lm} determined below so that

$$\Phi^{(k)}(su) = 0 \quad \text{for } 0 \leqslant k \leqslant K,\ s \in \mathcal{O}'(S). \tag{3}$$

There are standard polynomials A_s, B_s such that $\wp(sz) = A_s(\wp(z))/B_s(\wp(z))$, and we use these to define "denominators" for our auxiliary function as

$$\Delta_s(z) = \{B_s(\wp(z))B_s(\wp(\beta z))\}^D.$$

We also use the notation $\Theta_s(z) = \Delta_s(z)\Phi(sz)$. Then if we express each monomial of $\Theta_s(z)$ as

$$\Gamma_{slm}(z) = \Delta_s(z)(\wp(sz))^l(\wp(\beta sz))^m,$$

we have

$$\Theta_s(z) = \sum_{l=0}^{D} \sum_{m=0}^{D} P_{lm}(\wp(\beta u))\Gamma_{slm}(z).$$

By the Anderson-Baker-Coates Lemma [**5**, Lemma 6.2.3] there exist polynomials $G^k_{slm}(X, Y)$ satisfying

$$\Gamma^{(k)}_{slm}(u) = G^k_{slm}(\wp(\beta u), \wp'(\beta u)).$$

In this context $\deg_{\wp(\beta u)} G^k_{slm} \leqslant DS^2$ and $\deg_{\wp'(\beta u)} G^k_{slm} \leqslant 1$; and the coefficients of G^k_{slm} are polynomial expressions in $\wp(u)$ (of degree $\leqslant DS^2$), in $\wp'(u)$ (of degree $\leqslant 1$), in β (of degree $\leqslant K$) with coefficients integers in $\mathbf{Q}(g_2, g_3)$ with F-heights $< C_3(DS^2 + K \log K)$.

Setting

$$Q^k_s(X, Y) = \sum_{l=0}^{D} \sum_{m=0}^{D} P_{lm}(X) G^k_{slm}(X, Y)$$

for $0 \leqslant k \leqslant K$, $s \in \mathcal{O}'(S)$, we then take δ to be a denominator for all of $\wp(u)$, $\wp'(u)$, and β and consider the system of equations

$$\delta^{DS^2+K} Q^k_s(X, Y) = 0 \quad \text{for } 0 \leqslant k \leqslant K \text{ and } s \in \mathcal{O}'(S).$$

In this situation the coefficients of our unknowns are algebraic integers and we can therefore obtain solutions $P_{lm}(X) \in Z[X]$. Further, since we have $(D+1)^2$ unknowns and $(K+1)(S+1)^2$ equations, we can solve this system with $P_{lm}(X)$ satisfying $\deg P_{lm}(X) \leqslant DS^2$ and $\log \mathrm{ht}(P_{lm}) \leqslant C_5(DS^2 + K \log K)$. Therefore a simple computation yields

$$\deg_X Q^k_s(X, Y) \leqslant C_6 DS^2, \qquad \deg_Y Q^k_s(X, Y) \leqslant 1,$$

and

$$h_F\big(Q^k_s(X, Y)\big) \leqslant C_6\big(DS^2 + K \log K\big).$$

Notice that, by construction, $Q^k_S(\wp(u), \wp(\beta u)) = \Theta^{(k)}_s(u)$; then, since $\Delta_s(u) \neq 0$, we deduce that $\Phi^{(k)}(su) = 0$ $(0 \leqslant k \leqslant K, s \in \mathcal{O}'(S))$. (It is immediate that $\Delta_s(u) \neq 0$ by our assumption that u is a nontorsion point for $\wp$.)

The function $T(z) = (\sigma(z)\sigma(\beta z))^{C_6 DS^2}\Phi(z)$ is entire, therefore Schwarz's Lemma applied to circles of radii $r = CS$ and $R = 4CS$ allows us to majorize $T^{(k)}(su)$ as $\log|T^{(k)}(su)| \leqslant -C_7 KS^2$ for $0 \leqslant k < C^4 K$ and $s \in \mathcal{O}'(S)$. Then, by Lemma 2 we have a lower bound of the form

$$\log\Big|(d/dz)^k\big[\sigma(z)\sigma(\beta z)\big]^{C_6 DS^2}\Big|_{z=u} \geqslant -C_8 DS^4$$

and, hence, we may conclude that

$$\log|\Phi^{(k)}(su)| \leqslant -C_9 KS^2 \qquad \big(0 \leqslant k < C^4 K, s \in \mathcal{O}'(S)\big).$$

From this we estimate $|Q_s^k(\wp(\beta u), \wp'(\beta u))|$ by observing that

$$Q_s^k(\wp(\beta u), \wp'(\beta u)) = (d/dz)^k(\Delta_s(z)\Phi(sz))_{z=u},$$

and that for $0 \leqslant k < C^4K$, $\log|\Delta_s^k(u)| \leqslant C_{10}K\log K$. This, together with the usual estimate for the binomial coefficients, yields

$$\log|Q_s^k(\wp(\beta u), \wp'(\beta u))| \leqslant -C_{11}KS^2 \quad \text{for } 0 \leqslant k < C^4K \text{ and } s \in \mathcal{C}'(S).$$

If the polynomials $P_{lm}(X)$ have a common factor, $Q(X) \in Z[X]$, then we rewrite our auxiliary function as

$$\Phi(z) = Q(\wp(\beta u)) \sum_{l,m} \bar{P}_{lm}(\wp(\beta u))(\wp(z))^l(\wp(\beta z))^m,$$

where the polynomials $\bar{P}_{lm}(X)$ have no common factor.

By Lemma II, p. 135 of Gelfond **[2]** and the fact that $Q(X) \in Z[X]$ and, therefore $\mathrm{ht}(Q) \geqslant 1$, we have the estimates $\deg \bar{P}_{lm} \leqslant C_{12}DS^2$ and $\log \mathrm{ht}\, \bar{P}_{lm} \leqslant C_{12}(DS^2 + K\log K)$. Since these estimates are essentially the same as for the polynomials $P_{lm}(X)$ we may as well assume that the polynomials $P_{lm}(X)$ have no common factor.

Naturally $Q_s^k(\wp(\beta u), \wp'(\beta u))$ is possibly linear in $\wp'(\beta u)$. To avoid any complications arising from this fact we eliminate $\wp'(\beta u)$ by taking the relative norm of our expression from $F(\wp(\beta u), \wp'(\beta u))$ to $F(\wp(\beta u))$. These new expressions $\bar{Q}_s^k(\wp(\beta u))$ satisfy the same estimates with new constants.

Next, suppose that one of the polynomials $\bar{Q}_s^k(X)$ is prime to $P(X)$; we show below that this must be the case. Forming the resultant of these two polynomials with respect to X and applying Lemma 5, we obtain

$$\log|P(\wp(\beta u))| > -cdK\log K;$$

recalling the definitions of our parameters, we are done.

If, however, none of the polynomials $\bar{Q}_s^k(X)$ is prime to $P(X)$, then our assumption that $P(X)$ is irreducible implies that $P(X)$ must divide each $\bar{Q}_s^k$. Then choose $\theta \in C$ such that $P(\theta) = 0$, and $\alpha \in C$ such that $\wp(\alpha) = \theta$. Notice that $\bar{Q}_s^k(\theta) = 0$ for all k and s as above.

We now exhibit functions $\Phi_s(z)$ which violate the zeros estimate of Lemma 3. Let

$$\Phi_s(z) = \sum_{l=0}^{D} \sum_{m=0}^{D} P_{lm}(\wp(\alpha))(\wp(z))^l(\wp(\beta z + \delta_s))^m$$

where $\delta_1 = \alpha - \beta u$ and $\delta_s = s\delta_1$. Our goal is to show that $\Phi_s^{(k)}(su) = 0$ for $0 \leqslant k \leqslant C^3K$.

To see this, set

$$\bar{\Delta}_s(z) = \{B_s(\wp(z))B_s(\wp(\beta z + \delta_s))\}^D$$

and

$$\bar{\Gamma}_{slm}(z) = \bar{\Delta}_s(z)(\wp(sz))^l(\wp(s\beta z + \delta_s))^m.$$

Then if $\Theta_s^{(k)}(z) = \bar{\Delta}_s(z)\Phi_s(sz)$, we have the simple relationship:

$$\Theta_s(u) = \sum_{l=0}^{D} \sum_{m=0}^{D} P_{lm}(\wp(\alpha))\bar{\Gamma}_{slm}(u).$$

From this we next show that $\Theta_s^{(k)}(u) = 0$ for $0 \leqslant k \leqslant C^3K$ and $s \in \mathcal{O}'(S)$.

As in the construction of our auxiliary function we use the polynomials G_{slm}^k to express the derivatives of $\bar{\Gamma}_{slm}^{(k)}(z)$, that is, $\bar{\Gamma}_{slm}^{(k)}(u) = G_{slm}^k(\wp(\alpha u), \wp'(\alpha u))$. Therefore $\bar{Q}_s^k(\wp(\alpha)) = \Theta_s^{(k)}(u)$, which implies that

$$\Phi_s^{(k)}(su) = 0 \quad \text{for } 0 \leqslant k \leqslant C^3K \text{ and } s \in \mathcal{O}'(S). \tag{4}$$

The functions $\Phi_s(z)$ are of the type appearing in Lemma 3, therefore (provided that none of them vanishes identically) the total order of zeros at the points su, $s \in \mathcal{O}'(S)$, is bounded by $8(D + 1) + \frac{2}{3}DS^2$. However, (4) implies that we have at least $\frac{1}{3}C^3KS^2$ zeros (counting multiplicities) at these points. For our choice of parameters these bounds are contradictory.

Therefore unless some $\Phi_s(z)$ vanishes identically, $P(X)$ must be prime to one of the polynomials $\bar{Q}_s^k(X)$ and our lower bound holds. If one of the functions $\Phi_s(z)$ vanishes identically, then we have an algebraic dependence between the functions $\wp(z)$ and $\wp(\beta z + \delta_s)$, and hence between $\wp(z)$ and $\wp(\beta z)$. Therefore there exists a nonzero integer t such that $t\beta\Omega \subseteq \Omega$, and therefore $t\beta\omega_2 = n\omega_1 + m\omega_2$. This implies that β is linear over K_τ, contrary to our hypothesis.

We have demonstrated the desired minorization for $|P(\wp(\beta u))|$ when $P(X)$ is irreducible over $F[X]$. For the general case take $P(X) \in Z[X]$ and put $Q(X) = a^{-1}P(X)$ where a is the leading coefficient of $P(X)$. We then factor $Q(X)$ over $F[X]$ as $Q(X) = \prod_{i=0}^{k} P_i(X)$ where each polynomial $P_i(X)$ is nonconstant, monic and irreducible. If we let $d_i = \deg P$ and $t_i = (d_i + h_F(P_i))$, then by our first result,

$$\log|P_i(\wp(\beta u))| \geqslant -Cd_i^2t_i^2(\log t_i)^4 \quad \text{for } 1 \leqslant i \leqslant k.$$

Hence

$$|\log P(\wp(\beta u))| \geqslant \log|Q(\wp(\beta u))| > -C\sum_{i=0}^{k} d_i^2t_i^2(\log t_i)^4,$$

and by the usual theory of polynomial heights (e.g. Lang [**3**, p. 57]),

$$\sum_{i=0}^{k} d_i^2t_i^2(\log t_i)^4 \leqslant Cd^2t^2(\log t)^4.$$

This completes our proof of result A.

REMARKS ON THE PROOF OF THEOREM B. Without the hypothesis of complex multiplication we have fewer multiplications with which to extrapolate. In this case, the size of the zeros with which are extrapolate must be enlarged accordingly. This is achieved through the following choice of parameters: $D = [C^{13}d^3t(1 + \log t)^7]$, $S = [C^3d(1 + \log t)]$, $K = [C^{22}d^5t^2(1 + \log t)^{13}]$. Using these the proof of B is similar to that of A; therefore, we omit the proof of this result.

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