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THE CARTAN MATRIX OF A GROUP ALGEBRA MODULO ANY POWER OF ITS RADICAL

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ABSTRACT. We prove that the Cartan matrix of a group algebra $F[G]$ modulo any power of its radical J is dual symmetric, provided F is a splitting field of $F[G]/J$. This eases the process of determining the Loewy series of the projective indecomposable $F[G]$-modules.

Let G be any finite group and F a field of characteristic p. By a module of the group algebra $F[G]$, we will always mean a right module. Let $S_1, S_2, \ldots, S_k$ be a complete set of representatives of the isomorphism classes of simple $F[G]$-modules, and let P_i be the projective cover of S_i. For any $F[G]$-module M, we let M^* denote the dual $F[G]$-module of M. The dual of S_i is denoted by S_{i^*} as well. Thus $P_i^* = P_{i^*}$, as $F[G]$ is symmetric. In fact we will only use this fundamental property of a group algebra in the following. Consequently, similar results hold for symmetric algebras rather than just group algebras.

In the following we set $A = F[G]$ and denote its radical by J. As usual, the Loewy length of an A-module M is the minimal number r for which $MJ^r = 0$. Observe that M and M^* always have the same Loewy length, as $M = M^{**}$.

From now on we must assume F is a splitting field.

Now let I be any power of J and set $\bar{A} = A/I$, $\bar{P}_i = P_i/P_iI$. As $S_iI = 0$ for all i, $\{S_1, S_2, \ldots, S_k\}$ is also a complete set of representatives of isomorphism classes of simple $\bar{A}$-modules of the Artinian algebra $\bar{A}$. Recall that the Cartan matrix $\{c_{ij}^{-}\}$ of $\bar{A}$ is defined as follows: c_{ij}^{-} equals the multiplicity of S_j as a composition factor of $\bar{P}_i$, which conveniently equals $\dim(\mathrm{Hom}_{\bar{A}}(\bar{P}_j, \bar{P}_i))$ (see **[1]**). It is also well known that the Cartan matrix of A is symmetric. The aim of this note is to prove

THEOREM A. *The Cartan matrix $\{c_{ij}^{-}\}$ of $\bar{A}$ is dual symmetric, i.e.* $c_{ij}^{-} = c_{j^*i^*}^{-}$.

Our proof easily follows from the following elementary, but fundamental, fact.

LEMMA. *Let $\{\beta_1, \ldots, \beta_m\}$ be a basis of a complement to* $\mathrm{Hom}_A(P_i/P_iJ^{n-1}, P_j)$ *in* $\mathrm{Hom}_A(P_i/P_iJ^n, P_j)$. *Set*

$$E_i/P_iJ^n = \bigcap_{r=1}^{m} \ker \beta_r, \qquad E_j = \sum_{r=1}^{m} \operatorname{Im} \beta_r.$$

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Then

$$m = \dim(\mathrm{Hom}_A(S_j, P_i/E_i)) = \dim(\mathrm{Hom}_A(E_j, S_i)).$$

PROOF. The first equality follows from the fact that

$$m = \dim(\mathrm{Hom}_A(S_j, P_iJ^{n-1}/P_iJ^n)).$$

Moreover, for any $\beta = \Sigma\lambda_r\beta_r$, $\beta(P_i)$ is of Loewy length n. In particular, if π is the canonical homomorphism $E_j \to E_j/E_jJ$, then $\pi\beta_1,\ldots,\pi\beta_m$ are linearly independent, i.e. $E_j/E_jJ \simeq (S_i)^m$, which is equivalent to the second equality.

REMARK. It immediately follows from this observation that the Cartan matrix of A is symmetric. Indeed, if we denote the dimension above by m_n, we see that $c_{ij} = \Sigma_n m_n = c_{ji}$, where the first equality follows by considering the Loewy series of P_i, the second by considering the socle series of P_j.

THEOREM B. *The multiplicity of S_j in P_iJ^{n-1}/P_iJ^n equals that of S_{i^*} in $P_{j^*}J^{n-1}/P_{j^*}J^n$ for any n.*

PROOF. Using duality, it suffices to prove that the first number, m_1, is less than or equal to the second, m_2. By the Lemma,

$$m_1 = \dim(\mathrm{Hom}_A(S_j, P_i/P_iJ^n)) = \dim(\mathrm{Hom}_A(B, S_i)),$$

where B is a submodule of P_j of the form $B = \Sigma_{r=1}^m B_r$, and each B_r is a homomorphic image of P_i in P_j of Loewy length exactly n. But this implies that B^* is a quotient module of P_{j^*} with $B^*J^{n-1} = (S_{i^*})^{m_1}$.

Now, in general, if $0 \to L \to M \to N \to 0$ is exact, we obtain

$$L + MJ^{n-1}/L + MJ^n \simeq (L + MJ^{n-1}/L)/(L + MJ^n/L) \simeq NJ^{n-1}/NJ^n.$$

Choosing $M = P_{j^*}$ and $N = B^*$, we consequently obtain that, indeed, $m_1 \leqslant m_2$.

PROOF OF THEOREM A. We denote the Cartan matrix of A/J^n by $(c_{ij}(n))$ and use induction on n. Thus it suffices to prove that

$$c_{ij}(n) - c_{ij}(n-1) = c_{j^*i^*}(n) - c_{j^*i^*}(n-1),$$

which is exactly the statement of Theorem B.

This result has several applications, of which we just mention one:

COROLLARY. *Let $P_{i_1},\ldots,P_{i_e}$ be the projective modules of a self-dual block of $F[G]$. Assume that P_{i_1} is self-dual and the Loewy series of $P_{i_2},\ldots,P_{i_e}$ are known. Then the Loewy series of P_{i_1} is known as well, except for the composition factors isomorphic to S_{i_1}.*

REFERENCES

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