THE CARTAN MATRIX OF A GROUP ALGEBRA
MODULO ANY POWER OF ITS RADICAL

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Abstract. We prove that the Cartan matrix of a group algebra \( F[G] \) modulo any power of its radical \( J \) is dual symmetric, provided \( F \) is a splitting field of \( F[G]/J \). This eases the process of determining the Loewy series of the projective indecomposable \( F[G] \)-modules.

Let \( G \) be any finite group and \( F \) a field of characteristic \( p \). By a module of the group algebra \( F[G] \), we will always mean a right module. Let \( S_1, S_2, \ldots, S_k \) be a complete set of representatives of the isomorphism classes of simple \( F[G] \)-modules, and let \( P_i \) be the projective cover of \( S_i \). For any \( F[G] \)-module \( M \), we let \( M^* \) denote the dual \( F[G] \)-module of \( M \). The dual of \( S_i \) is denoted by \( S_i^* \), as well. Thus \( P_i^* = P_i \), as \( F[G] \) is symmetric. In fact we will only use this fundamental property of a group algebra in the following. Consequently, similar results hold for symmetric algebras rather than just group algebras.

In the following we set \( A = F[G] \) and denote its radical by \( J \). As usual, the Loewy length of an \( A \)-module \( M \) is the minimal number \( r \) for which \( MJ^r = 0 \). Observe that \( M \) and \( M^* \) always have the same Loewy length, as \( M = M^{**} \).

From now on we must assume \( F \) is a splitting field.

Now let \( I \) be any power of \( J \) and set \( \tilde{A} = A/I, \tilde{P_i} = P_i/P_i I \). As \( S_i I = 0 \) for all \( i \), \( \{ S_1, S_2, \ldots, S_k \} \) is also a complete set of representatives of isomorphism classes of simple \( \tilde{A} \)-modules of the Artinian algebra \( \tilde{A} \). Recall that the Cartan matrix \( \{ c_{ij} \} \) of \( \tilde{A} \) is defined as follows: \( c_{ij} \) equals the multiplicity of \( S_j \) as a composition factor of \( \tilde{P_i} \), which conveniently equals \( \dim(\text{Hom}_A(\tilde{P_i}, \tilde{P_j})) \) (see [1]). It is also well known that the Cartan matrix of \( A \) is symmetric. The aim of this note is to prove

**Theorem A.** The Cartan matrix \( \{ c_{ij} \} \) of \( \tilde{A} \) is dual symmetric, i.e. \( c_{ij} = c_{ji}^* \).

Our proof easily follows from the following elementary, but fundamental, fact.

**Lemma.** Let \( \{ \beta_1, \ldots, \beta_m \} \) be a basis of a complement to \( \text{Hom}_A(P_i/P_i J^{n-1}, P_j) \) in \( \text{Hom}_A(P_i/P_i J^n, P_j) \). Set

\[
E_i/P_i J^n = \bigcap_{r=1}^m \ker \beta_r, \quad E_j = \sum_{r=1}^m \text{Im} \beta_r.
\]

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205
Then

\[ m = \dim(\text{Hom}_A(S_j, P_i/E_i)) = \dim(\text{Hom}_A(E_j, S_i)). \]

**Proof.** The first equality follows from the fact that

\[ m = \dim(\text{Hom}_A(S_j, P_i J^{n-1}/P_i J^n)). \]

Moreover, for any \( \beta = \sum \lambda_\beta P_\beta \), \( \beta(P_i) \) is of Loewy length \( n \). In particular, if \( \pi \) is the canonical homomorphism \( E_j \rightarrow E_j/E_j J \), then \( \pi \beta_1, \ldots, \pi \beta_m \) are linearly independent, i.e. \( E_j/E_j J \cong (S_j)^m \), which is equivalent to the second equality.

**Remark.** It immediately follows from this observation that the Cartan matrix of \( A \) is symmetric. Indeed, if we denote the dimension above by \( m_n \), we see that

\[ c_{ij} = \sum m_n = c_{ji}, \]

where the first equality follows by considering the Loewy series of \( P_i \), the second by considering the socle series of \( P_j \).

**Theorem B.** The multiplicity of \( S_j \) in \( P_i J^{n-1}/P_i J^n \) equals that of \( S_j \) in \( P_j J^{n-1}/P_j J^n \) for any \( n \).

**Proof.** Using duality, it suffices to prove that the first number, \( m_1 \), is less than or equal to the second, \( m_2 \). By the Lemma,

\[ m_1 = \dim(\text{Hom}_A(S_j, P_i/P_i J^n)) = \dim(\text{Hom}_A(B, S_i)), \]

where \( B \) is a submodule of \( P_j \) of the form \( B = \sum_{r=1}^m B_r \), and each \( B_r \) is a homomorphic image of \( P_i \) in \( P_j \) of Loewy length exactly \( n \). But this implies that \( B^* \) is a quotient module of \( P_j \), with \( B^* J^{n-1} = (S_j)^m \).

Now, in general, if \( 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \) is exact, we obtain

\[ L + MJ^{n-1}/L + MJ^n \cong (L + MJ^{n-1}/L)/(L + MJ^n/L) \cong NJ^{n-1}/NJ^n. \]

Choosing \( M = P_j \) and \( N = B^* \), we consequently obtain that, indeed, \( m_1 \leq m_2 \).

**Proof of Theorem A.** We denote the Cartan matrix of \( A/J^n \) by \( (c_{ij}(n)) \) and use induction on \( n \). Thus it suffices to prove that

\[ c_{ij}(n) - c_{ij}(n-1) = c_{ji^*}(n) - c_{ji^*}(n-1), \]

which is exactly the statement of Theorem B.

This result has several applications, of which we just mention one:

**Corollary.** Let \( P_{i_1}, \ldots, P_{i_k} \) be the projective modules of a self-dual block of \( F[G] \). Assume that \( P_{i_1} \) is self-dual and the Loewy series of \( P_{i_2}, \ldots, P_{i_k} \) are known. Then the Loewy series of \( P_{i_1} \) is known as well, except for the composition factors isomorphic to \( S_{i_1} \).

**References**


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