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ON RELATIVE NORMAL COMPLEMENTS IN FINITE GROUPS. II

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ABSTRACT. Given a finite group G and subgroups H and H_0 with $H_0 \lhd H$, we let π denote the set of prime divisors of $(H: H_0)$, and we denote this configuration by (G, H, H_0, π). Pamela Ferguson has shown that if H/H_0 is solvable, then under certain conditions there exists a unique relative normal complement G_0 of H over H_0. In this paper we give alternative proofs of her two theorems.

Let G be a finite group and π a set of primes. The complementary set of primes will be denoted by π'. Let $\pi(G)$ denote the set of prime divisors of $|G|$. We call G a *π-group* if $\pi(G) \subseteq \pi$. An element x of G is a *π-element* if $\langle x \rangle$ is a π-group. Every element x of G has a unique decomposition, $x = x_\pi x_{\pi'} = x_{\pi'} x_\pi$, into a π-element x_π and a π'-element $x_{\pi'}$. Both x_π and $x_{\pi'}$ are powers of x. Two elements x and y belong to the same *π-section* of G if their π-parts x_π and y_π are conjugate in G. If S is a subset of G we let $S^{G,\pi}$ denote the union of all π-sections of G that intersect S.

We let (G, H, H_0, π) denote the following configuration. Let G be a finite group with subgroups H and H_0 such that $H_0 \triangle H$ and $\pi = \pi(H/H_0)$. Given this, and given a subgroup G_0 of G, G_0 is called a *relative normal complement* of H over H_0 if $G_0 \triangle G$, $G = G_0 H$, and $H_0 = G_0 \cap H$.

For (G, H, H_0, π) we consider the following conditions:

(A) Any two π-elements of $H - H_0$ that are G-conjugate are H-conjugate.

(B_0) For each π-element of $H - H_0$ we have $C_G(x) = O_{\pi'}(C_G(x))C_H(x)$.

(C) $|(H - H_0)^{G,\pi}| = (G: H)|H - H_0|$.

Pamela Ferguson has proved the following two theorems [**1**].

THEOREM 1. *If (G, H, H_0, π) satisfies conditions* (B_0) *and* (C) *and H/H_0 is solvable, then there exists a unique relative normal complement G_0 of H over H_0 and $G_0 = G - (H - H_0)^{G,\pi}$.*

THEOREM 2. *If (G, H, H_0, π) satisfies conditions* (A) *and* (B_0) *and H/H_0 is solvable, then there exists a unique relative normal complement G_0 of H over H_0 and $G_0 = G - (H - H_0)^{G,\pi}$.*

Theorem 2 was proved by Reynolds [**3**, Theorem 2] in the case that H/H_0 is a p-group.

It is the purpose of this paper to give alternative proofs of these theorems. First we prove a lemma to the effect that the two theorems are equivalent.

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LEMMA. *If* (G, H, H_0, π) *satisfies condition* (B_0), *then condition* (A) *is satisfied if and only if condition* (C) *is satisfied.*

PROOF. Suppose $x \in H - H_0$ and $y \in H$, and suppose x_π and y_π are H-conjugate. Since H/H_0 is a π-group, $x_{\pi'}$ and $y_{\pi'} \in H_0$ and $x_\pi \in H - H_0$. Hence, $y_\pi \in H - H_0$, so $y \in H - H_0$. Thus $H - H_0$ is a union of π-sections of H.

Condition (A) holds if and only if, for every π-section S of G, either $S \cap (H - H_0) = \varnothing$ or $S \cap (H - H_0)$ is a π-section of H. Hence, [**2**, Lemma 3.4] states that condition (A) holds if and only if condition (C) holds (assuming condition (B_0)), and our proof is complete.

Now we prove Theorem 2 (and hence Theorem 1).

PROOF OF THEOREM 2. Let (G, H, H_0, π) be given satisfying the hypotheses of Theorem 2. We may assume that the theorem holds for groups of order less than $|G|$. Since H/H_0 is solvable, there is a prime $p \in \pi$ such that H has a normal subgroup H_1 of index p with $H_1 \supseteq H_0$.

It is easily verified that $H - H_1$ is a union of p-sections of H. If x and y are G-conjugate p-elements of $H - H_1$, then, by hypothesis, they are H-conjugate. Thus $(G, H, H_1, \{p\})$ satisfies condition (A). If x is a p-element of $H - H_1$, then, by hypothesis,

$$C_G(x) = O_{\pi'}(C_G(x))C_H(x).$$

But $p \in \pi$, so

$$C_G(x) = O_{p'}(C_G(x))C_H(x).$$

Thus $(G, H, H_1, \{p\})$ satisfies condition (B_0). Therefore Reynolds' theorem [**3**, Theorem 2] implies that in G there is a normal complement G_1 of H over H_1.

Let $\pi_1 = \pi(H_1/H_0)$. We now verify that (G_1, H_1, H_0, π_1) satisfies the hypotheses of Theorem 2. Let x and y be G_1-conjugate π_1-elements of $H_1 - H_0$. By hypothesis they must be H-conjugate. Hence, there exist $g_1 \in G_1$ and $h \in H$ such that $y = x^{g_1} = x^h$. Hence, $g_1h^{-1} \in C_G(x)$, and hypothesis (B_0) implies there exist $g_2 \in O_{\pi'}(C_G(x))$ and $k \in C_H(x)$ such that $g_1h^{-1} = g_2k$. But $O_{\pi'}(C_G(x)) \subseteq G_1$ since G_1 is a normal subgroup of G of index p. Therefore $g_2^{-1}g_1 = kh \in G_1 \cap H = H_1$, so $g_2^{-1}g_1 = h_1 \in H_1$ for some h_1. Since $g_2 \in C_G(x)$, we have $y = x^{g_1} = x^{g_2^{-1}g_1} = x^{h_1}$. Thus x and y are H_1-conjugate, so (G_1, H_1, H_0, π) satisfies condition (A).

Let x be a π_1-element of $H_1 - H_0$. Since $\pi_1 \subseteq \pi$ and G_1 is a normal subgroup of G of index p, $O_{\pi'}(C_G(x))$ is a normal π_1'-subgroup of $C_{G_1}(x)$. Therefore hypothesis (B_0) implies

$$\begin{aligned} C_{G_1}(x) &= O_{\pi'}(C_G(x))C_H(x) \cap G_1 \\ &= O_{\pi'}(C_G(x))(C_H(x) \cap G_1) = O_{\pi'}(C_G(x))C_{H_1}(x), \end{aligned}$$

so that $(G_1, H_1 H_0, \pi_1)$ satisfies condition (B_0). Hence, by our induction hypothesis, there is a relative normal complement G_0 in G_1 of H_1 over H_0, and $G_0 = G_1 - (H_1 - H_0)^{G_1, \pi_1}$.

Since $G = G_1H$, $G_1 \triangle G$, and $H_1 - H_0$ is a normal subset of H, it is easily seen that $(H_1 - H_0)^{G_1,\pi_1}$ is a normal subset of G and, hence, that $G_0 \triangle G$. We have

$$G = G_1H, \quad G_1 \cap H = H_1, \quad \text{and} \quad G_1 = G_0H_1, \quad G_0 \cap H_1 = H_0.$$

Therefore

$$G = G_1H = G_0H_1H = G_0H$$

and

$$G_0 \cap H = G_0 \cap G_1 \cap H = G_0 \cap H_1 = H_0,$$

so G_0 is a relative normal complement in G of H over H_0.

It remains to show that $G_0 = G - (H - H_0)^{G,\pi}$. According to the above lemma,

$$|(H - H_0)^{G,\pi}| = (G : H)\,|H - H_0|,$$

so

$$|G - (H - H_0)^{G,\pi}| = (G : H)\,|H_0|.$$

Therefore [**2**, Proposition 2.2] yields our equation for G_0, and the proof of the theorem is complete.

References

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