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ON TENSOR PRODUCTS AND EXTENDED CENTROIDS[1]

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ABSTRACT. For prime algebras R and S over a field F it is shown that each nonzero ideal of $R \otimes S$ contains a nonzero element $r \otimes s$, $r \in R$, $s \in S$, if and only if $C(R) \otimes C(S)$ is a field, where $C(R)$ (respectively $C(S)$) is the extended centroid of R (respectively S).

A prime F-algebra R is said to be closed (or centrally closed) if F coincides with the extended centroid of R. This terminology arises in **[2]** where it is noted **[2,** Remark p. 60] that if R is a closed prime unital F-algebra and S is any unital F-algebra, then any nonzero ideal of $R \otimes_F S$ contains a nonzero element of the form $r \otimes s$, $r \in R$, $s \in S$. The aim of this note is to show that this condition is satisfied if and only if $C(R) \otimes_F C(S)$ is a field, where $C(R)$ and $C(S)$ are the extended centroids of the prime algebras R and S, respectively, which are not necessarily unital. Observe that when R is closed $C(R) = F$ and $C(R) \otimes_F C(S) = C(S)$ is a field. In addition to the notation already introduced we write $R' = RC(R)$ for the central closure of R. All tensor products will be over F so the subscript F will be omitted. To deal with not necessarily unital rings note that a prime F-algebra R can be embedded as an ideal in a prime F-algebra $R^{\#}$ with unity and so $C(R) = C(R^{\#})$. Throughout this paper R (and S) are prime associative algebras so $C(R)$ is a field.

LEMMA 1 **[4,** THEOREM 1]. *Let $x, y \in R'$ such that $xry = yrx$ for all $r \in R$. Then x and y are $C(R)$-dependent.*

LEMMA 2. *If $\mu(x \otimes y) = 0$ for $x \in R'$, $y \in S'$ and $\mu \in C(R) \otimes C(S)$, then either $\mu = 0$ or $x \otimes y = 0$.*

PROOF. If $\mu \neq 0$ write $\mu = \sum_{i=1}^{m} c_i \otimes d_i$ where $\{c_i\}$ is an F-independent set. If $x \neq 0$ then $\{c_i x\}$ is also F-independent (since $C(R)$ is a field) so $\mu(x \otimes y) = 0$ implies $d_i y = 0$ for all i, whence $y = 0$ and $x \otimes y = 0$. $\square$

THEOREM. *Let R and S be prime F-algebras. Then the following are equivalent:*
(1) *$C(R) \otimes C(S)$ is a field;*
(2) *each nonzero ideal of $R \otimes S$ contains an element $r \otimes s \neq 0$, $r \in R$, $s \in S$.*

PROOF. We first prove the result when R and S are unital.

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(1) $\Rightarrow$ (2). Let $0 \neq I \lhd R \otimes S$ and let $0 \neq x = \sum_{i=1}^{m} r_i \otimes s_i \in I$ where m is minimal among all such representations of nonzero elements of I. Then $\{s_i\}$ is F-independent and, for all $r \in R$, $y = x(rr_1 \otimes 1) - (r_1 r \otimes 1)x \in I$. The minimality of m forces $y = 0$ and so $r_i r r_1 = r_1 r r_i$ for all i. Since r_1 and r_i are nonzero $r_i = c_i r_1$ for some $c_i \in C(R)$ by Lemma 1. Similarly $s_i = d_i s_1$ for some $d_i \in C(S)$. Put $\lambda = \sum_{i=1}^{m} c_i \otimes d_i \in C(R) \otimes C(S)$. Thus $x = \lambda(r_1 \otimes s_1)$ and $\lambda \neq 0$. By (1) $r_1 \otimes s_1 = \mu x$ for $\mu = \lambda^{-1} \in C(R) \otimes C(S)$.

Write $\mu = \sum_{j=1}^{n} e_j \otimes f_j$ with $e_j \in C(R), f_j \in C(S)$. From the construction of R' and S' there are nonzero ideals A of R, B of S such that $Ae_j \subseteq R$ and $Bf_j \subseteq S$ for all j. Hence $(A \otimes B)\mu \subseteq R \otimes S$ and $(A \otimes B)(r_1 \otimes s_1) \subseteq I$. Since R and S are prime we have $0 \neq r \in Ar_1$ and $0 \neq s \in Bs_1$ so that $r \otimes s \in I$.

(2) $\Rightarrow$ (1). Let $0 \neq \mu \in C(R) \otimes C(S)$. As above we have $0 \neq A \lhd R$, $0 \neq B \lhd S$ and $\mu(A \otimes B) = (A \otimes B)\mu = I \lhd R \otimes S$. From Lemma 2, $I \neq 0$ and so I contains a length one tensor from (2). Select $\sum_{i=1}^{m} a_i \otimes b_i \in A \otimes B$ with m as small as possible so that $\mu(\sum a_i \otimes b_i)$ is of length one, say $\mu(\sum a_i \otimes b_i) = r_1 \otimes s_1 \neq 0$, $r_1 \in R$, $s_1 \in S$. Then, for all $r \in R$ and $i = 1, 2, \ldots, m$, $(a_i r \otimes 1)(r_1 \otimes s_1) - (r_1 \otimes s_1)(ra_i \otimes 1) \in I$. The minimality of m along with Lemma 1 gives $a_i = p_i r_1$, $p_i \in C(R)$. Similarly $b_i = q_i s_1$ for $q_i \in C(S)$. With $\lambda = \sum p_i \otimes q_i \in C(R) \otimes C(S)$ we have

$$(1 - \mu\lambda)(r_1 \otimes s_1) = 0.$$

By Lemma 2, $1 = \mu\lambda$ and $C(R) \otimes C(S)$ is a field.

To deal with the not necessarily unital case we use the algebras $R^{\#}$ and $S^{\#}$.

(1) $\Rightarrow$ (2). If $0 \neq I \lhd R \otimes S$, then $0 \neq (R^{\#} \otimes S^{\#})I(R^{\#} \otimes S^{\#}) = J \lhd R^{\#} \otimes S^{\#}$ and $J \subseteq I$. Now J contains an element $r' \otimes s' \neq 0$ with $r' \in R^{\#}$ and $s' \in S^{\#}$. Since $(r' \otimes s')(R \otimes S) \subseteq I$ and R and S are prime, we have $0 \neq r \otimes s \in I$ for some $r \in R$, $s \in S$.

(2) $\Rightarrow$ (1). It is clear that condition (2) implies that $R \otimes S$ is prime. Hence Theorem 2.3 of [3] implies that $R^{\#} \otimes S^{\#}$ is prime. If $0 \neq J \lhd R^{\#} \otimes S^{\#}$ then $0 \neq J(R \otimes S) = I \lhd R \otimes S$ contains a length one tensor $r \otimes s$, $r \in R$, $s \in S$. Thus $R^{\#} \otimes S^{\#}$ satisfies (2) and $C(R) \otimes C(S)$ is a field. □

REMARK. Condition (1) of Theorem 1 is equivalent to the statement that $C(R) \otimes C(S)$ is a domain and $C(R) \otimes C(S) = C(R \otimes S)$. If these properties are possessed by $C(R) \otimes C(S)$ then $R \otimes S$ is prime by [3, Theorem 2.3] and $C(R \otimes S)$ is a field, whence $C(R) \otimes C(S)$ is a field. For the converse we appeal to a recent theorem of Matczuk [5] which says that if $R \otimes S$ is semiprime, then

$$C(R \otimes S) = C(C(R) \otimes C(S))$$

(here the extended centroid is defined as in the more general setting of Amitsur [1]). Thus if $C(R) \otimes C(S)$ is a field we have $C(R \otimes S) = C(R) \otimes C(S)$.

COROLLARY 1. *If R and S are simple F-algebras such that $C(R) \otimes C(S)$ is a field, then $R \otimes S$ is a simple F-algebra.*

Resco [6] has shown that if R is a closed primitive F-algebra and S is a primitive F-algebra, then $R \otimes S$ is primitive. His proof depends on property (2) and so can be easily modified to prove the next corollary.

COROLLARY 2. *If R and S are primitive F-algebras such that $C(R) \otimes C(S)$ is a field, then $R \otimes S$ is primitive.*

PROOF. If R and S are unital then the argument from [**6**] applies. Otherwise we take the rings $R^{\#}$ and $S^{\#}$, which are now primitive, and then $R^{\#} \otimes S^{\#}$ is primitive and so $R \otimes S$ is primitive. □

It is clear that if the hypotheses of Corollary 2 hold then $R' \otimes S'$ is primitive. However this conclusion follows from $R \otimes S$ being primitive, as we show in the next proposition, which extends the known result that if R is primitive then so is R'.

PROPOSITION. *Let R and S be prime F-algebras such that $R \otimes S$ is primitive. Then $R' \otimes S'$ is primitive.*

PROOF. Assume first that R and S are unital. Let L be a left ideal of $R \otimes S$ such that $(R \otimes S)/L$ is a faithful irreducible $R \otimes S$-module. Put $L' = (R' \otimes S')L$.

If $1 = \Sigma \lambda_i l_i$, $\lambda_i \in R' \otimes S'$, $l_i \in L$, then there are $0 \neq A \lhd R$, $0 \neq B \lhd S$ such that $(A \otimes B)\lambda_i \subseteq R \otimes S$ and so $0 \neq A \otimes B \subseteq L$. But $(R \otimes S)/L$ is faithful so we have a contradiction and $L' \neq R' \otimes S'$.

Since $R \otimes S$ is prime, $R' \otimes S'$ is prime by [**3**, Theorem 2.3]. If $0 \neq X' \lhd R' \otimes S'$ and $0 \neq x' \in X'$, then there are ideals $0 \neq I \lhd R$, $0 \neq J \lhd S$ such that $(I \otimes J)x' \subseteq R \otimes S$. Now $0 \neq (C(R) \otimes C(S))(I \otimes J) \lhd R' \otimes S'$, so the primeness of $R' \otimes S'$ implies that $(I \otimes J)x' \neq 0$. Thus $X = (R \otimes S) \cap X' \neq 0$ and $X \lhd R \otimes S$. Now $L + X = R \otimes S$ so $L' + (R' \otimes S')X = R' \otimes S'$. Hence $L' + X' = R' \otimes S'$. Therefore, by the comaximality criterion for primitivity, we have that $R' \otimes S'$ is primitive.

In the general case the algebra $R^{\#} \otimes S^{\#}$ is prime from [**3**, Theorem 2.3], and so primitive since it contains $R \otimes S$ as an ideal. Thus $(R^{\#})' \otimes (S^{\#})'$ is primitive. However $R' \lhd (R^{\#})'$ and $S' \lhd (S^{\#})'$ so $R' \otimes S'$ is primitive. □

REFERENCES

1. S. A. Amitsur, *On rings of quotients*, Symposia Mathematica **8** (1972), 149–164.
2. T. S. Erickson, W. S. Martindale III and J. M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975), 49–63.
3. J. Krempa, *On semisimplicity of tensor products*, (Proc. Antwerp Conf., 1978), Pure and Appl. Math., vol. 51, Dekker, New York, 1979, pp. 105–122.
4. W. S. Martindale III, *Prime rings satisfying a generalized polynomial identity*, J. Algebra **12** (1969), 576–584.
5. J. Matczuk, *Central closure of semiprime tensor products*, Comm. Algebra **10** (1982), 263–278.
6. R. Resco, *A reduction theorem for the primitivity of tensor product*, Math. Z. **170** (1980), 65–76.

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