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THE STRUCTURE OF THE AUTOMORPHISM GROUP OF A FREE GROUP ON TWO GENERATORS

DRAGOMIR Ž. ĐOKOVIĆ[1]

ABSTRACT. Let $F_2 = Z * Z$ be a free group of rank two. We show that Aut F_2 can be built up from cyclic groups by using only the free products and semidirect products. Explicitly we have Aut $F_2 = ((Z * Z) \rtimes (Z_3 * Z_3)) \rtimes (Z_4 \rtimes Z_2)$. As a corollary we obtain a simple presentation of Aut F_2.

Let $F_2 = Z * Z = \langle x, y \rangle$ be a free group of rank 2 with generators x and y, $Z_3 * Z_3 = \langle a, b : a^3 = b^3 = 1 \rangle$ the free product of two cyclic groups of order 3, and $D_4 = Z_4 \rtimes Z_2 = \langle c, d : c^4 = d^2 = (cd)^2 = 1 \rangle$ the dihedral group of order 8. (We use $*$ (resp. $\rtimes$) to denote the free (resp. semidirect) product of two groups.) Let $\alpha, \beta \in$ Aut $F_2 = \Phi$ be defined by

$$\alpha: x \to x^{-1}y^{-1}, \qquad y \to x; \tag{1}$$

$$\beta: x \to y^{-1}, \qquad y \to xy^{-1}. \tag{2}$$

Since $\alpha^3 = \beta^3 = 1$ we have a homomorphism $Z_3 * Z_3 \to \Phi$ sending $a \to \alpha$ and $b \to \beta$. Let $H = F_2 \rtimes (Z_3 * Z_3)$ be the corresponding semidirect product. Thus H is generated by x, y, a, b with defining relations

$$a^3 = b^3 = 1, \qquad axa^{-1} = x^{-1}y^{-1}, \qquad aya^{-1} = x, \qquad bxb^{-1} = y^{-1}, \qquad byb^{-1} = xy^{-1}. \tag{3}$$

Using (3) it is easy to verify that H has endomorphisms γ and δ such that

$$\gamma: x \to y^{-1}, \qquad y \to x, \qquad a \to b, \qquad b \to xa; \tag{4}$$

$$\delta: x \to y, \qquad y \to x, \qquad a \to y^{-1}a^{-1}, \qquad b \to b^{-1}. \tag{5}$$

From (4) and (5) one derives that $\gamma^4 = \delta^2 = (\gamma\delta)^2 = 1$, and so there is a homomorphism $D_4 \to$ Aut H sending $c \to \gamma$ and $d \to \delta$. We let

$$G = H \rtimes D_4 = (F_2 \rtimes (Z_3 * Z_3)) \rtimes D_4$$

be the corresponding semidirect product. Thus G is generated by x, y, a, b, c, d with defining relations those of H and D_4 together with

$$cxc^{-1} = y^{-1}, \qquad cyc^{-1} = x, \qquad cac^{-1} = b, \qquad cbc^{-1} = xa, \qquad dxd = y, \qquad dad = y^{-1}a^{-1}, \qquad dbd = b^{-1}. \tag{6}$$

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Clearly $F_2 \lhd G$ and let ϕ: $G \to \Phi$ be the canonical homomorphism: $\phi(z)$ is the restriction to F_2 of the inner automorphism $t \to ztz^{-1}$ of G.

THEOREM. *ϕ is an isomorphism.*

PROOF. We have $F_2 \lhd G$ and $F_2 \lhd \Phi$ and the restriction of ϕ to F_2 is the identity map. Hence it suffices to show that the induced map $\bar{\phi}$: $\bar{G} \to \bar{\Phi}$ ($\bar{G} = G/F_2$ and $\bar{\Phi} = \Phi/F_2$) is an isomorphism. Let $\bar{F_2} = F_2/F_2'$ where F_2' is the commutator subgroup of F_2. The group $\bar{F_2}$ is free abelian of rank 2 with a basis $\{\bar{x}, \bar{y}\}$. One knows [**2**, Proposition 4.5, p. 25] that the canonical homomorphism ψ: $\bar{\Phi} \to \operatorname{Aut} \bar{F_2}$ is an isomorphism. By using ψ and the basis $\{\bar{x}, \bar{y}\}$ of $\bar{F_2}$ we shall identify $\bar{\Phi}$ with $\mathrm{GL}_2(Z)$. From (3) and (6) we deduce that

$$(7) \qquad \bar{\phi}(\bar{b}) = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \bar{\phi}(\bar{c}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \bar{\phi}(\bar{d}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Now we need the fact that $\mathrm{GL}_2(Z)$ is generated by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

with defining relations (see [**2**, p. 25] or [**1**, Chapter 7])

$$(8) \qquad A^6 = (AC)^2 = (BC)^2 = A^3B^2 = 1.$$

It follows from the definition of G and $\bar{G}$ that

$$(9) \qquad \bar{b}^3 = \bar{c}^4 = (\bar{b}\bar{d})^2 = (\bar{b}\bar{c}^2)^2 = 1.$$

Using (8) and (9) one can easily check that there is a homomorphism θ: $\mathrm{GL}_2(Z) \to \bar{G}$ such that

$$(10) \qquad \theta: A \to \bar{b}\bar{c}^2, \quad B \to \bar{c}^{-1}, \quad C \to \bar{d}.$$

Since $\bar{G}$ is generated by $\bar{b}, \bar{c}$ and $\bar{d}$ it follows from (7) and (10) that $\bar{\phi}$ and θ are inverses of each other.

As a corollary we can obtain a simple presentation of Φ. For that purpose we shall first "simplify" the above presentation of G by eliminating the redundant generators x, y and b. Let K be the group with generators u, v, w and defining relations

$$(11) \qquad u^3 = v^4 = w^2 = (vw)^2 = v^2uv^2wuw = [uvu, v^2] = 1.$$

LEMMA. *There is an isomorphism f: $K \to G$ such that*

$$(12) \qquad f: u \to a, \quad v \to c, \quad w \to d.$$

PROOF. In G we have $a^3 = c^4 = d^2 = (cd)^2 = 1$, $c^2ac^2dad = \gamma^2(a)\delta(a) = xa \cdot y^{-1}a^{-1} = x\alpha(y^{-1}) = 1$, and

$$[aca, c^2] = acac^2a^{-1}c^{-1}a^{-1}c^2 = a\gamma(a)\gamma^{-1}(a^{-1})\gamma^2(a^{-1})$$
$$= a \cdot b \cdot b^{-1}y \cdot a^{-1}x^{-1} = \alpha(y)x^{-1} = 1.$$

This proves that there exists a homomorphism f: $K \to G$ such that (12) holds.

Similar routine (but longer) computations show that there is a homomorphism g: $G \to K$ such that

$$(13) \qquad x \to v^2uv^2u^{-1}, \quad y \to (u^{-1}w)^2, \quad a \to u, \quad b \to vuv^{-1}, \quad c \to v, \quad d \to w.$$

From (12) and (13) it follows that f and g are inverses of each other.

COROLLARY. *Let* ε (*resp.* ζ) *be the restriction of* γ (*resp.* δ) *to* F_2. *Then* Φ *is generated by* α, ε, ζ *with defining relations*

$$\alpha^3 = \varepsilon^4 = \zeta^2 = (\varepsilon\zeta)^2 = \varepsilon^2\alpha\varepsilon^2\zeta\alpha\zeta = [\alpha\varepsilon\alpha, \varepsilon^2] = 1.$$

PROOF. This follows from the theorem, the lemma, and the fact that the isomorphism $\phi \circ f$: $K \to \Phi$ sends $u \to \alpha$, $v \to \varepsilon$, $w \to \zeta$.

In conclusion let us mention that another simple presentation of Φ is due to B. H. Neumann [**4**], see also [**3**, p. 169].

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DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF WATERLOO, WATERLOO, ONTARIO, N2L 3G1, CANADA