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ULTIMATELY CLOSED PROJECTIVE RESOLUTIONS AND RATIONALITY OF POINCARÉ-BETTI SERIES

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ABSTRACT. A condition on the syzygies of a module is given which implies the rationality of certain Poincaré-Betti series.

Let A be a finite-dimensional algebra over a field F. All modules considered here will be finitely generated left A-modules, and we will write $|M|$ for $\dim_F M$.

Given an ordered pair (M, N) of A-modules, the Poincaré-Betti series for (M, N) is the formal power series $\sum_{k=0}^{\infty}(-T)^k |\operatorname{Ext}^k(M, N)|$. In certain special cases, it has been conjectured that these series are rational. The most notable such special case is when A is a commutative, local, noetherian algebra (not necessarily finite dimensional) and $M = N = F$, where F is given the trivial module structure. Despite many results showing rationality, Anick [**1**] has given an example of an algebra A where the Poincaré-Betti series for (F, F) is not rational. Roos [**4**] gives an excellent survey of results in this direction.

For finite-dimensional algebras, we give a condition which implies the rationality of certain Poincaré-Betti series. We also provide some information about when this condition is satisfied.

Let M be a finitely generated A-module with the minimal projective resolution

$$\cdots \xrightarrow{d_2} Q_1 \xrightarrow{d_1} Q_0 \xrightarrow{d_0} M \to 0.$$

As usual, we denote the kth syzygy module by $\Omega^k M = \operatorname{im} d_k = \operatorname{Ker} d_{k-1}$.

DEFINITION (JANS [**3**]). Let M be a finitely generated A-module. We say that M has an ultimately closed projective resolution if there is an $n > 0$ such that each indecomposable summand of $\Omega^n M$ is already a summand of $\Omega^k M$ for some $k < n$.

Since for finite-dimensional algebras A, every finitely generated A-module decomposes uniquely into a direct sum of indecomposable modules, one can see immediately that M has an ultimately closed resolution if and only if it satisfies the following condition.

($*$) The set $\{X \in \operatorname{mod} A\colon X \text{ is an indecomposable summand of } \Omega^k M,\ k \geq 0\}$ has only finitely many isomorphism classes of modules.

We will use this formulation in proving the following

THEOREM. *Let A be a finite-dimensional algebra and M an indecomposable module with an ultimately closed projective resolution. Then for any A-module Y, the Poincaré-Betti series for (M, Y), $\sum_{k=0}^{\infty}(-T)^k |\operatorname{Ext}^k(M, Y)|$, is rational.*

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PROOF. Let $S = \{M = N_1, \dots, N_L\}$ be a set of representatives of the isomorphism classes of the modules in $\{X \in \text{mod } A\colon X \text{ is an indecomposable summand of } \Omega^k M$ for some $k \geqslant 0\}$. We notice that $\Omega^n(\Omega^j M) = \Omega^{n+j} M$, so all summands of $\Omega^n N_i$ are also in S. In particular, all summands of $\Omega^1 N_i$ are in S. Now for each N_i, write the beginning of a projective resolution

$$0 \to \sum_{j=1}^{L} a_{ji} N_j \to P(N_i) \to N_i \to 0$$

where $a_{ji} \geqslant 0$. We see that for all $k > 1$ and any module Y,

$$\begin{aligned}\operatorname{Ext}^k(N_i, Y) &= \operatorname{Ext}^{k-1}(\Omega N_i, Y) = \operatorname{Ext}^{k-1}\left(\sum_{j=1}^{L} a_{ji} N_j, Y\right)\\ &= \sum_{j=1}^{L} a_{ji} \operatorname{Ext}^{k-1}(N_j, Y).\end{aligned}$$

Writing $r_i = \sum_{k=0}^{\infty} (-T)^k \,|\operatorname{Ext}^k(N_i, Y)|$, this formula shows that for each i,

$$\begin{aligned} r_i &= |\operatorname{Ext}^0(N_i, Y)| - T|\operatorname{Ext}^1(N_i, Y)| + \sum_{k=2}^{\infty} (-T)^k |\operatorname{Ext}^k(N_i, Y)| \\ &= |\operatorname{Hom}(N_i, Y)| - T|\operatorname{Ext}^1(N_i, Y)| + \sum_{k=2}^{\infty} (-T)^k \left(\sum_{j=1}^{L} a_{ji} |\operatorname{Ext}^{k-1}(N_j, Y)| \right) \\ &= |\operatorname{Hom}(N_i, Y)| - T|\operatorname{Ext}^1(N_i, Y)| - T\sum_{j=1}^{L} a_{ji} \left(\sum_{m=1}^{\infty} (-T)^m |\operatorname{Ext}^m(N_j, Y)| \right) \\ &= |\operatorname{Hom}(N_i, Y)| - T|\operatorname{Ext}^1(N_i, Y)| - T\sum_{j=1}^{L} a_{ji} \big(r_j - |\operatorname{Hom}(N_j, Y)| \big). \end{aligned}$$

Thus, we have

$$\sum_{j=1}^{L} (\delta_{ji} + Ta_{ji}) r_j = |\operatorname{Hom}(N_i, Y)| - T|\operatorname{Ext}(N_i, Y)| + T\sum_{j=1}^{L} a_{ji} |\operatorname{Hom}(N_j, Y)|\,.$$

Let $\mu \in M_L(\mathbf{Z}[T])$ be the matrix $(\delta_{ji} + Ta_{ji})$. Direct computation shows that $\det \mu = 1 + Tp(T)$ for some $p \in \mathbf{Z}[T]$, so μ is invertible over the integral power series ring $\mathbf{Z}[\![T]\!]$. Further, Cramer's rule shows that μ^{-1} has rational coordinates. In $(\mathbf{Z}[\![T]\!])^L$ we have the equation $(r_1, \dots, r_L)\mu = (p_1, \dots, p_L)$, where

$$p_i = |\operatorname{Hom}(N_i, Y)| - T|\operatorname{Ext}(N_i, Y)| + T\sum_{j=1}^{L} a_{ji} |\operatorname{Hom}(N_j, Y)|\,.$$

Multiplying both sides by μ^{-1}, we see that the vector $(r_1, \dots, r_L) = (p_1, \dots, p_L)\mu^{-1}$ has rational coordinates. In particular, $r_1 = \sum_{k=0}^{\infty} (-T)^k \,|\operatorname{Ext}^k(M, Y)|$ is rational, as claimed.

In order to apply the Theorem, we need some examples of modules with ultimately closed projective resolutions. First, we recall the definition [**2**] of stable equivalence.

The stable module category $\underline{\text{mod}}\, A$ has as its objects the modules without projective summands. For modules X and Y without projective summands, we write $P(X, Y)$ for the group of maps factoring through a projective module. We then define the morphisms in $\underline{\text{mod}}\, A$ by $\underline{\text{Hom}}(X, Y) = \text{Hom}(X, Y)/P(X, Y)$. Given two algebras A and B, we say that A is stably equivalent to B if the categories $\underline{\text{mod}}\, A$ and $\underline{\text{mod}}\, B$ are equivalent.

PROPOSITION. *In the following cases, all A-modules have ultimately closed projective resolutions.* (i) *A has finite representation type, i.e. there are only finitely many nonisomorphic indecomposable A-modules.*

(ii) $\text{rad}^2 A = 0$,

(iii) *A is stably equivalent to a hereditary algebra.*

(iv) *There are only finitely many nonisomorphic indecomposable, torsionless A-modules.*

(v) *The set $S_k = \{X \in \text{mod}\, A$: X is an indecomposable summand of $\Omega^k M$ for some $M\}$ has only finitely many isomorphism classes for some $k \geqslant 0$.*

PROOF. We begin by noting the following dependences among the conditions: (i) $\Rightarrow$ (iv) $\Rightarrow$ (v) and (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v).

That (i) implies (iv) is trivial. Any summand of a syzygy $\Omega^k M$ is torsionless, so (iv) implies (v). Auslander and Reiten [**2**] show that (ii) implies (iii) and also that, for any A stably equivalent to a hereditary algebra, every indecomposable torsionless module is either simple or projective, hence (iii) implies (iv). Clearly, it suffices to show that if A satisfies (v), all A-modules have ultimately closed projective resolutions.

Assume that A satisfies (v). Given any module M, there are only finitely many indecomposable summands of $\Omega^0 M, \Omega^1 M, \ldots, \Omega^{k-1} M$. For $i \geqslant k$, $\Omega^i M = \Omega^k(\Omega^{i-k} M)$ so all summands of $\Omega^i M$ are in the finite set S_k. Hence, M satisfies condition $(*)$ and must have an ultimately closed projective resolution.

COROLLARY. *In cases* (i)–(v), *the Poincaré-Betti series* $\sum_{k=0}^{\infty}(-T)^k \mid \text{Ext}^k(M, N) \mid$ *is rational for all M and N.*

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