ON THE AUTOMORPHISM GROUP
OF A LINEAR ALGEBRAIC MONOID

MOHAN S. PUTCHA

Abstract. Let $S$ be a connected regular monoid with zero. It is shown that an
automorphism of $S$ is inner if and only if it sends each idempotent of $S$ to a
conjugate idempotent. In the language of semigroup theory, the automorphism
group of $S$ maps homomorphically into the automorphism group of the finite lattice
of $J$-classes of $S$, and the kernel of this homomorphism is the group of inner
automorphisms of $S$. In particular, if the $J$-classes of $S$ are linearly ordered, then
every automorphism of $S$ is inner.

Throughout this paper $Z^+$ will denote the set of all positive integers and $K$ an
algebraically closed field. $\mathbb{M}_n(K)$ denotes the monoid of all $n \times n$ matrices over $K$. $GL(n, K)$ denotes the group of units of $\mathbb{M}_n(K)$, and $SL(n, K)$ the group of
matrices of determinant 1 in $\mathbb{M}_n(K)$. We will follow the notation and terminology
of [2, 4] concerning linear algebraic monoids. Let $S$ be a connected algebraic monoid
with group of units $G$. By an automorphism of $S$ is meant a semigroup automorphism
$\sigma$ of $S$ such that both $\sigma$ and $\sigma^{-1}$ are polynomial maps. An automorphism $\sigma$ of $S$ is
inner if there exists $g \in G$ such that $\sigma(a) = g^{-1}ag$ for all $a \in S$. We let $\mathcal{R}(S)$ denote
the finite lattice of all regular $J$-classes of $S$, and $E(S)$ the partially ordered set of all
idempotents of $S$. It follows from the work of the author [4, 5] and Renner [7, 8] that
$S$ is regular if and only if the closure of the radical of $G$ is a Clifford semigroup. In
particular, if $S$ has a zero then $S$ is regular if and only if $G$ is a reductive group.

Theorem 1. Let $S$ be a connected regular monoid with zero and $\sigma$ an automorphism
of $S$. Then $\sigma$ is an inner automorphism of $S$ if and only if $\sigma(J) = J$ for all $J \in \mathcal{R}(S)$
(i.e. $\sigma(e)$ is a conjugate of $e$ for all $e \in E(S)$).

Proof. Suppose $\sigma(J)$ for all $J \in \mathcal{R}(S)$. We must show that $\sigma$ is inner. Let $G$
denote the group of units of $S$, and let $T$ be a maximal torus of $G$. Suppose first that
$T = G$. Then $\sigma(e) = e$ for all $e \in E(T)$. We prove by induction on dim $T$ that $\sigma$
is the identity map. First suppose that dim $T = 1$. Then by [1, Exercise 4, p. 57] either
$\sigma(t) = t$ for all $t \in T$, or else $\sigma(t) = t^{-1}$ for all $t \in T$. In the latter case $t\sigma(t) = 1$
for all $t \in T$ and, hence, for all $t \in \overline{T}$. Since $0 \in \overline{T}$, this is a contradiction. So let
dim $T > 1$. Let $F = \{t \in T | \sigma(t) = t\}^c$. Let $e \in E(\overline{T})$, $e \neq 0$. Let
\[ T_e = \{a \in T | ae = e\}^c. \]
Since $\sigma(e) = e$, $\sigma(T_e) = T_e$. Since $e$ is the zero of $\overline{T}$, we see by the induction hypothesis that $T_e \subseteq F$. Thus $e \in \overline{F}$. So $E(\overline{T}) \setminus \{0\} \subseteq \overline{F}$. There exists $f \in E(\overline{T})$, $f \neq 1, 0$. So [2, Theorem 1.4] there exists $h \in E(\overline{T})$, $h \neq 0$, such that $fh = 0$. Since $f$, $h \in \overline{F}$, $0 \in \overline{F}$. So $E(\overline{T}) = E(\overline{F})$. By [2, Theorem 1.4], dim $T = \dim F$. Thus $T = F$ and $\sigma$ is the identity map. In particular, the automorphism group of $\overline{T}$ is isomorphic to a subgroup of the group of automorphisms of $E(\overline{T})$ and, hence, is a finite group.

Let us now consider the general case. Since all maximal tori of $G$ are conjugate, we can assume without loss of generality that $\sigma(T) = T$. Let $\Lambda \subseteq E(\overline{T})$ be a cross-section lattice of $S$ (see [6]). In other words, if $e, f \in \Lambda$, $f \in \mathcal{S} \mathcal{S}$, then $e \geq f$, each idempotent of $S$ is conjugate to an idempotent in $\Lambda$, and no two idempotents in $\Lambda$ are conjugate. So $\sigma(\Lambda) \subseteq E(\overline{T})$ is also a cross-section lattice of $S$. By [6, Theorem 11] there exists $u \in W$, the Weyl group of $G$ relative to $T$, such that $\sigma(\Lambda) = u^{-1} \Lambda u$.

Thus, without loss of generality, we can assume that $\sigma(\Lambda) = \Lambda$. Let $e \in \Lambda$. Then $e$, $\sigma(e) \in \Lambda$. By hypothesis, $e \not< \sigma(e)$. Since $\Lambda$ is a cross-section lattice of $S$, $e = \sigma(e)$. So $\sigma(e) = e$ for all $e \in \Lambda$. Let $F = \{t \in T| \sigma(t) = t\}$. Since the automorphism group of $\overline{T}$ is finite, we see that there exists $k \in Z^+$ such that $\sigma^k(a) = a$ for all $a \in \overline{T}$. So by [3, Lemma 1.13], $\Lambda \subseteq \overline{F}$. Since $\Lambda$ contains a maximal chain of $E(\overline{T})$, we see by [2, Theorem 1.4] that $\dim F = \dim T$, so $T = F$. Thus $\sigma(t) = t$ for all $t \in T$. Let $\Gamma$ be a maximal chain in $E(\overline{T})$, and let $B = \{a \in G| ae = eae \text{ for all } e \in \Gamma\}$. Since $\sigma(\Gamma) = \Gamma$, we see that $\sigma(B) = B$. Since $G$ is a reductive group, we see by [4, Theorem 4.5] that $B$ is a Borel subgroup of $G$. Clearly $T \subseteq B$. We see by [1, Theorem 27.4(b) and 9, Theorem 11.4.3] that $\sigma$ is an inner automorphism.

Example. Let $S$ denote the Zariski closure in $\mathbb{M}_3(K) \times \mathbb{M}_3(K)$ of $\{(\alpha A, \alpha (A^{-1})^T)| \alpha \in K, A \in SL(3, K)\}$. Then $S$ is a connected regular monoid with zero, $\mathbb{M}(S) = \{G, J, J_1, J_2, 0\}$ with $G > J > J_i > 0$, $i = 1, 2$. Let $e, e_1, e_2$, denote diagonal matrices with the respective diagonals being $(((1,0,0), (0,0,1)), ((1,0,0), (0,0,0)), ((0,0,0), (0,0,1)))$. Then $e \in J$, $e_1 \in J_1$, $e_2 \in J_2$ and $\{1, e, e_1, e_2, 0\}$ is a cross-section lattice of $S$. Let $\sigma: S \to S$ be given by $\sigma(A, B) = (B, A)$. Then $\sigma$ is an automorphism of $S$ which is not inner. Note that $\sigma$ induces a nontrivial automorphism of $\mathbb{M}(S)$: $\sigma(G) = G, \sigma(J) = J, \sigma(0) = 0, \sigma(J_1) = J_2, \sigma(J_2) = J_1$.

Theorem 1 can be restated as follows.

**Theorem 2.** Let $S$ be a connected regular monoid with zero. Then the automorphism group of $S$ maps homomorphically into the automorphism group of $\mathbb{M}(S)$. The kernel of this homomorphism is the group of inner automorphisms of $S$.

**Corollary.** Let $S$ be a connected regular monoid with zero such that the $\frac{q}{2}$-classes of $S$ are linearly ordered. Then every automorphism of $S$ is inner.

**Remark.** The above corollary applies to the multiplicative monoid $\mathbb{M}_n(K)$. Note that the map $A \to (A^{-1})^T$ is an automorphism of $GL(n, K)$ which is not inner.

**References**


DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27650