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ON THE AUTOMORPHISM GROUP OF A LINEAR ALGEBRAIC MONOID

MOHAN S. PUTCHA[1]

ABSTRACT. Let S be a connected regular monoid with zero. It is shown that an automorphism of S is inner if and only if it sends each idempotent of S to a conjugate idempotent. In the language of semigroup theory, the automorphism group of S maps homomorphically into the automorphism group of the finite lattice of $\mathcal{J}$-classes of S, and the kernel of this homomorphism is the group of inner automorphisms of S. In particular, if the $\mathcal{J}$-classes of S are linearly ordered, then every automorphism of S is inner.

Throughout this paper Z^+ will denote the set of all positive integers and K an algebraically closed field. $\mathfrak{M}_n(K)$ denotes the monoid of all $n \times n$ matrices over K. $\mathrm{GL}(n, K)$ denotes the group of units of $\mathfrak{M}_n(K)$, and $\mathrm{SL}(n, K)$ the group of matrices of determinant 1 in $\mathfrak{M}_n(K)$. We will follow the notation and terminology of **[2, 4]** concerning linear algebraic monoids. Let S be a connected algebraic monoid with group of units G. By an *automorphism* of S is meant a semigroup automorphism σ of S such that both σ and σ^{-1} are polynomial maps. An automorphism σ of S is *inner* if there exists $g \in G$ such that $\sigma(a) = g^{-1}ag$ for all $a \in S$. We let $\mathfrak{U}(S)$ denote the finite lattice of all regular $\mathcal{J}$-classes of S, and $E(S)$ the partially ordered set of all idempotents of S. It follows from the work of the author **[4, 5]** and Renner **[7, 8]** that S is regular if and only if the closure of the radical of G is a Clifford semigroup. In particular, if S has a zero then S is regular if and only if G is a reductive group.

THEOREM 1. *Let S be a connected regular monoid with zero and σ an automorphism of S. Then σ is an inner automorphism of S if and only if $\sigma(J) = J$ for all $J \in \mathfrak{U}(S)$ (i.e. $\sigma(e)$ is a conjugate of e for all $e \in E(S)$).*

PROOF. Suppose $\sigma(J)$ for all $J \in \mathfrak{U}(S)$. We must show that σ is inner. Let G denote the group of units of S, and let T be a maximal torus of G. Suppose first that $\overline{T} = G$. Then $\sigma(e) = e$ for all $e \in E(\overline{T})$. We prove by induction on $\dim T$ that σ is the identity map. First suppose that $\dim T = 1$. Then by **[1**, Exercise 4, p. 57**]** either $\sigma(t) = t$ for all $t \in T$, or else $\sigma(t) = t^{-1}$ for all $t \in T$. In the latter case $t\sigma(t) = 1$ for all $t \in T$ and, hence, for all $t \in \overline{T}$. Since $0 \in \overline{T}$, this is a contradiction. So let $\dim T > 1$. Let $F = \{t \in T \mid \sigma(t) = t\}^c$. Let $e \in E(\overline{T})$, $e \neq 0$. Let

$$T_e = \{a \in T \mid ae = e\}^c.$$

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Since $\sigma(e) = e$, $\sigma(T_e) = T_e$. Since e is the zero of $\bar{T}_e$, we see by the induction hypothesis that $T_e \subseteq F$. Thus $e \in \bar{F}$. So $E(\bar{T})\backslash\{0\} \subseteq \bar{F}$. There exists $f \in E(\bar{T})$, $f \neq 1, 0$. So [**2**, Theorem 1.4] there exists $h \in E(\bar{T})$, $h \neq 0$, such that $fh = 0$. Since f, $h \in \bar{F}$, $0 \in \bar{F}$. So $E(\bar{T}) = E(\bar{F})$. By [**2**, Theorem 1.4], $\dim T = \dim F$. Thus $T = F$ and σ is the identity map. In particular, the automorphism group of $\bar{T}$ is isomorphic to a subgroup of the group of automorphisms of $E(\bar{T})$ and, hence, is a finite group.

Let us now consider the general case. Since all maximal tori of G are conjugate, we can assume without loss of generality that $\sigma(T) = T$. Let $\Lambda \subseteq E(\bar{T})$ be a cross-section lattice of S (see [**6**]). In other words, if $e, f \in \Lambda$, $f \in SeS$, then $e \geqslant f$, each idempotent of S is conjugate to an idempotent in Λ, and no two idempotents in Λ are conjugate. So $\sigma(\Lambda) \subseteq E(\bar{T})$ is also a cross-secton lattice of S. By [**6**, Theorem 11] there exists $u \in W$, the Weyl group of G relative to T, such that $\sigma(\Lambda) = u^{-1}\Lambda u$. Thus, without loss of generality, we can assume that $\sigma(\Lambda) = \Lambda$. Let $e \in \Lambda$. Then e, $\sigma(e) \in \Lambda$. By hypothesis, $e \mathcal{J} \sigma(e)$. Since Λ is a cross-section lattice of S, $e = \sigma(e)$. So $\sigma(e) = e$ for all $e \in \Lambda$. Let $F = \{t \in T \mid \sigma(t) = t\}^c$. Since the automorphism group of $\bar{T}$ is finite, we see that there exists $k \in Z^+$ such that $\sigma^k(a) = a$ for all $a \in \bar{T}$. So by [**3**, Lemma 1.13], $\Lambda \subseteq \bar{F}$. Since Λ contains a maximal chain of $E(\bar{T})$, we see by [**2**, Theorem 1.4] that $\dim F = \dim T$, so $T = F$. Thus $\sigma(t) = t$ for all $t \in T$. Let Γ be a maximal chain in $E(\bar{T})$, and let $B = \{a \in G \mid ae = eae$ for all $e \in \Gamma\}$. Since $\sigma(\Gamma) = \Gamma$, we see that $\sigma(B) = B$. Since G is a reductive group, we see by [**4**, Theorem 4.5] that B is a Borel subgroup of G. Clearly $T \subseteq B$. We see by [**1**, Theorem 27.4(b) and **9**, Theorem 11.4.3] that σ is an inner automorphism.

EXAMPLE. Let S denote the Zariski closure in $\mathfrak{M}_3(K) \times \mathfrak{M}_3(K)$ of $\{(\alpha A, \alpha(A^{-1})^T) \mid \alpha \in K, A \in \mathrm{SL}(3, K)\}$. Then S is a connected regular monoid with zero, $\mathfrak{U}(S) = \{G, J, J_1, J_2, 0\}$ with $G > J > J_i > 0$, $i = 1, 2$. Let e, e_1, e_2, denote diagonal matrices with the respective diagonals being ((1, 0, 0), (0, 0, 1)), ((1, 0, 0), (0, 0, 0)), ((0, 0, 0), (0, 0, 1)). Then $e \in J$, $e_1 \in J_1$, $e_2 \in J_2$ and $\{1, e, e_1, e_2, 0\}$ is a cross-section lattice of S. Let σ: $S \to S$ be given by $\sigma(A, B) = (B, A)$. Then σ is an automorphism of S which is not inner. Note that σ induces a nontrivial automorphism of $\mathfrak{U}(S)$: $\sigma(G) = G$, $\sigma(J) = J$, $\sigma(0) = 0$, $\sigma(J_1) = J_2$, $\sigma(J_2) = J_1$.

Theorem 1 can be restated as follows.

THEOREM 2. *Let S be a connected regular monoid with zero. Then the automorphism group of S maps homomorphically into the automophism group of $\mathfrak{U}(S)$. The kernel of this homomorphism is the group of inner automorphisms of S.*

COROLLARY. *Let S be a connected regular monoid with zero such that the $\mathcal{J}$-classes of S are linearly ordered. Then every automorphism of S is inner.*

REMARK. The above corollary applies to the multiplicative monoid $\mathfrak{M}_n(K)$. Note that the map $A \to (A^{-1})^T$ is an automorphism of $\mathrm{GL}(n, K)$ which is not inner.

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DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, NORTH CAROLINA 27650