PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 2, June 1983

SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS HAVING ALL ZEROS INSIDE [-1, 1]

A. K. VARMA

ABSTRACT. Let H_n be the set of all algebraic polynomials whose degree is n and whose zero are all real and lie inside $[-1,1]$. Then for n even we have ($n = 2m$)

$$\int_{-1}^{1}(P_n(x))^2 \geq (n/2 + 3/4 + 3/4(n-1))\int_{-1}^{1}P_n^2(x)\,dx,$$

where equality holds iff $P_n(x) = (1-x^2)^m$. If n is an odd positive integer, a similar inequality is valid (see (1.6) below). In the case $P_n \in H_n$ and subject to the condition $P_n(1) = 1$, then

$$\int_{-1}^{1}(P_n'(x))^2\,dx \geq \frac{n}{4} + \frac{1}{8} + \frac{1}{8(2n-1)},$$

where equality holds for $P_n(x) = ((1+x)/2)^n$.

Let H_n be the set of all polynomials whose degree is n and whose zeros are all real and lie inside $[-1,1]$. In this work we are concerned with the following theorems of P. Turán [3], A. K. Varma [4], and J. Szabadös and A. K. Varma [2].

THEOREM A (P. TURÁN). *Let $P_n \in H_n$; then*

$$\max_{-1\leq x\leq 1}|P_n'(x)| > \left(\frac{n}{6}\right)^{1/2}\max_{-1\leq x\leq 1}|P_n(x)|. \tag{1.1}$$

($P_n'(x)$ stands for the derivative of $P_n(x)$.)

THEOREM B (A. K. VARMA). *Let $P_n \in H_n$; then*

$$\|P_n'\|^2_{L_2[-1,1]} > \left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4n}\right)\|P_n\|^2_{L_2[-1,1]}, \qquad n \geq 13, \tag{1.2}$$

where

$$\|P_n\|^2_{L_2[-1,1]} = \int_{-1}^{1}P_n^2(x)\,dx.$$

Further, if $P_n(1) = P_n(-1) = 0$ and $n \geq 2$, $P_n \in H_n$, then

$$\|P_n'\|^2_{L_2[-1,1]} \geq \left(\frac{n}{2}+\frac{3}{4}+\frac{3}{4(n-1)}\right)\|P_n\|^2_{L_2[-1,1]}, \tag{1.3}$$

where equality holds for $P_n(x) = (1-x^2)^m$, $n = 2m$.

Received by the editors April 22, 1982. This paper was presented at the 2nd Edmonton Conference on Approximation Theory held on June 6–11, 1982.

1980 *Mathematics Subject Classification.* Primary 41A17; Secondary 26C05, 26D10.

THEOREM C (J. SZABADÖS AND A. K. VARMA). *Let $P_n \in H_n$, subject to the condition $P_n(1) = 1$; then*

$$\|P_n'\|^2_{L_2[-1,1]} > n/4. \tag{1.4}$$

The object of this paper is to prove the following theorems:

THEOREM 1. *Let $P_n \in H_n$ and $n = 2m$; then*

$$\|P_n'\|^2_{L_2[-1,1]} \geqslant \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)}\right)\|P_n\|^2_{L_2[-1,1]} \tag{1.5}$$

where equality holds iff $P_n(x)(1 - x^2)^m$. Moreover, if $n = 2m - 1$, then

$$\|P_n'\|^2_{L_2[-1,1]} \geqslant \left(\frac{n}{2} + \frac{3}{4} + \frac{5}{4(n-2)}\right)\|P_n\|^2_{L_2[-1,1]}, \qquad n \geqslant 3, \tag{1.6}$$

where equality holds iff

$$P_n(x) = (1-x)^{m-1}(1+x)^m \quad or \quad P_n(x) = (1-x)^m(1+x)^{m-1}.$$

THEOREM 2. *Let $P_n \in H_n$, subject to the condition $P_n(1) = 1$; then*

$$\|P_n'\|^2_{L_2[-1,1]} \geqslant \frac{n}{4} + \frac{1}{8} + \frac{1}{8(2n-1)}, \qquad n \geqslant 1, \tag{1.7}$$

where equality holds for $P_n(x) = ((1 + x)/2)^n$.

REMARK. Theorem 1 is an improvement over Theorem B in two respects. First, the condition $P_n(1) = P_n(-1) = 0$ is not necessary for (1.3) to hold. Secondly, we have precise bounds for n even and for n odd as mentioned in (1.5) and (1.6). Also note that (1.7) is an improvement over (1.4). A more precise form of Theorem A is due to J. Eröd [**1**].

2. Preliminaries. The following results are known:

$$P_n'(x) = P_n(x)\sum_{k=1}^{n}\frac{1}{x - x_k}, \tag{2.1}$$

$$(P_n'(x))^2 - P_n(x)P_n''(x) = (P_n(x))^2\sum_{k=1}^{n}\frac{1}{(x-x_k)^2}, \tag*{(2.2)$'$}$$

where $x_1, x_2, \ldots, x_n$ denote the roots of $P_n(x)$ such that

$$-1 \leqslant x_n \leqslant x_{n-1} \leqslant \cdots \leqslant x_2 \leqslant x_1 \leqslant 1. \tag{2.3}$$

We also note that

$$\frac{(x_1 - x)(x - x_n)}{(x - x_k)^2} = 1 - \frac{2x}{x - x_k} + \frac{x_1 + x_n}{x - x_k} + \frac{(x_1 - x_k)(x_k - x_n)}{(x - x_k)^2}. \tag{2.4}$$

3. Some identities. For the proof of Theorem 1 the following identities are needed.

IDENTITY 3.1. Let

$$\alpha(x) = (x_1 - x)(x - x_n), \tag{3.1}$$

and let $P_n(x)$ be any polynomial of degree n having $x_1, x_2, \ldots, x_n$ as its roots. Then

$$(3.2)\quad 2\int_{x_n}^{x_1}(P_n'(x))^2\,dx = (n+1)\int_{x_n}^{x_1}\frac{P_n^2(x)}{\alpha(x)}\,dx + \lambda_0 + \frac{1}{2}\int_{x_n}^{x_1}\frac{P_n^2(x)(\alpha'(x))^2}{(\alpha(x))^2}\,dx,$$

where

$$(3.3)\quad \lambda_0 = \int_{x_n}^{x_1}\frac{P_n^2(x)}{\alpha(x)}\sum_{k=1}^{n}\frac{(x_1-x_k)(x_k-x_n)}{(x-x_k)^2}\,dx.$$

PROOF. On integrating by parts and using $P_n(x_1) = P_n(x_n) = 0$, we obtain

$$2\int_{x_n}^{x_1}(P_n'(x))^2\,dx = \int_{x_n}^{x_1}\left((P_n'(x))^2 - P_n(x)P_n''(x)\right)dx.$$

Next, on using (2.2) and (2.4), we obtain

$$(3.4)\quad 2\int_{x_n}^{x_1}(P_n'(x))^2\,dx = n\int_{x_n}^{x_1}\frac{P_n^2(x)}{\alpha(x)}\,dx + \lambda_0 + \int_{x_n}^{x_1}\frac{\alpha'(x)}{\alpha(x)}P_n(x)P_n'(x)\,dx.$$

But integrating by parts gives us

$$(3.5)\quad \int_{x_n}^{x_1}\frac{\alpha'(x)}{\alpha(x)}P_n(x)P_n'(x)\,dx = \int_{x_n}^{x_1}\frac{P_n^2(x)}{\alpha(x)}\,dx + \frac{1}{2}\int_{x_n}^{x_1}\frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)}\,dx.$$

From (3.4) and (3.5), (3.2) follows.

Next, we shall prove

IDENTITY 3.2. Let $P_n(x)$ be any polynomial of degree n having $x_1, x_2, \ldots, x_n$ as its roots; then

$$(3.6)\quad 2\int_{x_n}^{x_1}(P_n'(x))^2\alpha(x)\,dx = n\int_{x_n}^{x_1}P_n^2(x)\,dx + S_0,$$

where

$$(3.7)\quad S_0 = \int_{x_n}^{x_1}\sum_{k=1}^{n}\frac{(x_1-x_k)(x_k-x_n)}{(x-x_k)^2}P_n^2(x)\,dx,$$

and $\alpha(x)$ is as defined by (3.1).

PROOF. By integrating by parts, we have

$$(3.8)\quad \begin{aligned} 2\int_{x_n}^{x_1}(P_n'(x))^2\alpha(x)\,dx &= \int_{x_n}^{x_1}\left((P_n'(x))^2 - P_n(x)P_n''(x)\right)\alpha(x)\,dx \\ &\quad - \int_{x_n}^{x_1}P_n(x)P_n'(x)\alpha'(x)\,dx \\ &= \int_{x_n}^{x_1}P_n^2(x)\sum_{k=1}^{n}\frac{\alpha(x)}{(x-x_k)^2}\,dx - \int_{x_n}^{x_1}P_n^2(x)\sum_{k=1}^{n}\frac{\alpha'(x)}{x-x_k}\,dx \\ &= \int_{x_n}^{x_1}P_n^2(x)\sum_{k=1}^{n}\left(\frac{\alpha(x)}{(x-x_k)^2} - \frac{\alpha'(x)}{x-x_k}\right)dx. \end{aligned}$$

By using (2.4) and (3.7) we obtain (3.6).

Now, we turn to prove

IDENTITY 3.3. Let $P_n(x)$ be any polynomial of degree n having $x_1, x_2, \ldots, x_n$ as its zeros; then (for $n = 2m$)

$$(3.9) \quad I_n \equiv \int_{x_n}^{x_1} \left(\sqrt{\alpha(x)}\, P_n'(x) - \frac{m\alpha'(x)}{\sqrt{\alpha(x)}} P_n(x) \right)^2 dx$$

$$= -\frac{n}{2}(2n+1)\int_{x_n}^{x_1} P_n^2(x)\, dx + \frac{1}{2} S_0 + (x_1 - x_n)^2 m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)}\, dx,$$

where S_0 is defined by (3.7). Further,

$$(3.10) \quad I_n' \equiv \int_{x_n}^{x_1} \left(P_n'(x) - \frac{m\alpha'(x)}{\alpha(x)} P_n(x) \right)^2 dx$$

$$= \int_{x_n}^{x_1} (P_n'(x))^2\, dx + m(m-1)\int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)}\, dx - n\int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)}\, dx.$$

PROOF. From the definition of I_n and using identity (3.2), we have

$$I_n = \frac{n}{2}\int_{x_n}^{x_1} P_n^2(x)\, dx + \frac{1}{2} S_0 - n\int_{x_n}^{x_1} P_n^2(x)\, dx + m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha(x)}\, dx.$$

But

$$(\alpha'(x))^2 = (x_1 - x_n)^2 - 4\alpha(x).$$

Therefore

$$I_n = -\frac{n}{2}\int_{x_n}^{x_1} P_n^2(x)\, dx + \frac{1}{2} S_0 - n^2 \int_{x_n}^{x_1} P_n^2(x)\, dx + (x_1 - x_n)^2 m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)}\, dx.$$

this proves (3.9). Similarly from the definition of I_n' as given in (3.10), we have

$$(3.11) \quad I_n' = \int_{x_n}^{x_1} (P_n'(x))^2\, dx + m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)}\, dx$$

$$-n\int_{x_n}^{x_1} P_n(x) P_n'(x) \frac{\alpha'(x)}{\alpha(x)}\, dx.$$

But

$$(3.12) \quad \int_{x_n}^{x_1} P_n(x) P_n'(x) \frac{\alpha'(x)}{\alpha(x)}\, dx = -\frac{1}{2}\int_{x_n}^{x_1} \frac{P_n^2(x)\big(\alpha''(x)\alpha(x) - (\alpha'(x))^2\big)}{\alpha^2(x)}\, dx$$

$$= \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)}\, dx + \frac{1}{2}\int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)}\, dx.$$

From (3.11) and (3.12), (3.10) follows at once.

4. Proof of Theorem 1. Let $P_n \in H_n$, $P_n(x_1) = P_n(x_n) = 0$. Since $P_n(x)$ has no zeros inside $[x_1, 1]$,

$$\sum_{k=1}^{n} \frac{1}{x - x_k} - \frac{n}{2} = \sum_{k=1}^{n} \left(\frac{1}{x - x_k} - \frac{1}{2} \right) = \sum_{k=1}^{n} \frac{2 - (x - x_k)}{2(x - x_k)} > 0$$

for $x_1 \leqslant x \leqslant 1$. Therefore, using (2.1),

$$\text{(4.1)} \quad \int_{x_1}^{1} (P_n'(x))^2 \, dx = \int_{x_1}^{1} P_n^2(x) \left(\sum_{k=1}^{n} \frac{1}{(x - x_k)} \right)^2 dx \geqslant \frac{n^2}{4} \int_{x_1}^{1} P_n^2(x) \, dx.$$

Similarly,

$$\text{(4.2)} \quad \int_{-1}^{x_n} (P_n'(x))^2 \, dx \geqslant \frac{n^2}{4} \int_{-1}^{x_n} P_n^2(x) \, dx.$$

Next we consider two cases. First let $x_1 = x_n = x_k$, $k = 1, 2, \ldots, n$. Then in view of (4.1) and (4.2),

$$\text{(4.3)} \quad \int_{-1}^{1} (P_n'(x))^2 \, dx \geqslant \frac{n^2}{4} \int_{-1}^{1} P_n^2(x) \, dx$$
$$> \left(\frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)} \right) \int_{-1}^{1} P_n^2(x) \, dx \quad (\text{for } n \geqslant 3).$$

So we turn to the case when $x_1 \neq x_n$. On using (3.2) and (3.10) we obtain ($n = 2m$, $m > 1$)

$$2\int_{x_n}^{x_1} (P_n'(x))^2 \, dx = \lambda_0 + (n+1) \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} dx$$
$$+ \frac{1}{2m(m-1)} \left\{ I_n' + n \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} dx - \int_{x_n}^{x_1} (P_n'(x))^2 \, dx \right\}.$$

Since $I_n' \geqslant 0$ and equal to zero iff $P_n(x) = (1 - x^2)^m$, $n = 2m$, we can rewrite the above as

$$\frac{(n-1)^2}{n(m-1)} \int_{x_n}^{x_1} (P_n'(x))^2 \, dx \geqslant \frac{m(n-1)}{(m-1)} \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} dx + \lambda_0.$$

Next, using (3.9) and observing that $I_n \geqslant 0 = 0$ iff $P_n(x) = (1 - x^2)^m$, $n = 2m$, we obtain

$$\text{(4.4)} \quad \frac{(n-1)^2}{n(m-1)} \int_{x_n}^{x_1} (P_n'(x))^2 \, dx$$
$$\geqslant \lambda_0 + \frac{(n-1)}{m(m-1)(x_1 - x_n)^2} \left\{ -\frac{1}{2} S_0 + \frac{n(2n+1)}{2} \int_{x_n}^{x_1} P_n^2(x) \, dx \right\}.$$

But using (3.3), (3.7) and

$$\alpha(x) = (x_1 - x)(x - x_n) \leqslant (x_1 - x_n)^2, \qquad x_n \leqslant x \leqslant x_1,$$

we obtain

$$\lambda_0 \geqslant (x_1 - x_n)^{-2} S_0.$$

Therefore (4.4) becomes

$$\frac{(n-1)^2}{n(m-1)} \int_{x_n}^{x_1} (P_n'(x))^2\,dx \geqslant \frac{(2n^2-2n+1)}{n(m-1)} S_0 + \frac{(n-1)(2n+1)}{(m-1)(x_1-x_n)^2} \int_{x_n}^{x_1} P_n^2(x)\,dx.$$

Since $S_0 \geqslant 0$, $m > 1$, we have

$$\int_{x_n}^{x_1} (P_n'(x))^2\,dx \geqslant \frac{(2n+1)n}{(n-1)(x_1-x_n)^2} \int_{x_n}^{x_1} P_n^2(x)\,dx. \tag{4.5}$$

On using (4.1), (4.2) and (4.5) we obtain

$$\begin{aligned}\int_{-1}^{1} (P_n'(x))^2\,dx &\geqslant \frac{n^2}{4}\int_{-1}^{x_n} P_n^2(x)\,dx + \frac{n^2}{4}\int_{x_1}^{1} P_n^2(x)\,dx \\ &\quad + \frac{(2n+1)n}{(n-1)(x_1-x_n)^2}\int_{x_n}^{x_1} P_n^2(x)\,dx \\ &\geqslant \frac{(2n+1)n}{4(n-1)}\int_{-1}^{1} P_n^2(x)\,dx\end{aligned}$$

(for $n \geqslant 4$) where equality holds when $x_1 = 1$, $x_n = -1$. This proves Theorem 1 for n even. For n odd the proof of (1.6) is similar, so we omit the details. Further, for $n = 1, 2$, inequality (1.5) and (1.6) can be easily verified.

PROOF OF THEOREM 2. Let $P_n \in H_n$, $P_n(1) = 1$, and let the zeros of $P_n(x)$ be given by

$$-1 \leqslant x_n \leqslant x_{n-1} \leqslant \cdots \leqslant x_2 \leqslant x_1 < 1. \tag{4.4}$$

Since

$$\begin{aligned}\int_{x_1}^{1} (P_n'(x))^2\,dx &= \int_{x_1}^{1} \left(P_n'(x) - \frac{nP_n(x)}{x-x_n} \right)^2 dx \\ &\quad - n^2\int_{x_1}^{1} \frac{P_n^2(x)}{(x-x_n)^2}\,dx + 2n\int_{x_1}^{1} \frac{P_n(x)P_n'(x)}{x-x_n}\,dx.\end{aligned} \tag{4.5}$$

But, on integrating by parts,

$$\int_{x_1}^{1} \frac{2P_n(x)P_n'(x)}{x-x_n}\,dx = \frac{P_n^2(1)}{1-x_n} + \int_{x_1}^{1} \frac{P_n^2(x)}{(x-x_n)^2}\,dx. \tag{4.6}$$

Using (4.5) and (4.6) we obtain

$$\int_{x_1}^{1}(P_n'(x))^2\,dx=\int_{x_1}^{1}\left(P_n'(x)-\frac{nP_n(x)}{x-x_n}\right)^2dx+\frac{nP_n(1)}{(1-x_n)} \tag{4.7}$$

$$-n(n-1)\int_{x_1}^{1}\frac{P_n^2(x)}{(x-x_n)^2}\,dx.$$

Next we note that

$$\int_{x_1}^{1}(P_n'(x))^2\,dx=\int_{x_1}^{1}P_n^2(x)\left(\sum_{k=1}^{n}\frac{1}{x-x_k}\right)^2dx.$$

Further

$$x-x_k\leqslant x-x_n,\qquad k=1,2,\ldots,n;\ x_1\leqslant x\leqslant 1.$$

Therefore

$$\int_{x_1}^{1}(P_n'(x))^2\,dx\geqslant n^2\int_{x_1}^{1}\frac{P_n^2(x)}{(x-x_n)^2}\,dx. \tag{4.8}$$

Thus, from (4.7) and (4.8),

$$\int_{x_1}^{1}(P_n'(x))^2\,dx\geqslant\int_{x_1}^{1}\left(P_n'(x)-\frac{nP_n(x)}{x-x_n}\right)^2dx+\frac{nP_n^2(1)}{1-x_n}-\frac{(n-1)}{n}\int_{x_1}^{1}(P_n'(x))^2\,dx.$$

In other words,

$$\frac{(2n-1)}{n}\int_{x_1}^{1}(P_n'(x))^2\,dx\geqslant\frac{nP_n^2(1)}{1-x_n}+\int_{x_1}^{1}\left(P_n'(x)-\frac{nP_n(x)}{x-x_n}\right)^2dx.$$

From this we conclude that

$$\int_{x_1}^{1}(P_n'(x))^2\,dx\geqslant\frac{n^2}{(2n-1)(1-x_n)}P_n^2(1)\geqslant\frac{n^2}{2(2n-1)}P_n^2(1)$$

where equality holds iff $x_n=-1$ and $P_n'(x)-nP_n(x)/(x+1)=0$. But then $P_n(x)=(1+x)^n/2^n$ and all roots of $P_n(x)$ are equal to -1. From this we finally have

$$\int_{-1}^{1}(P_n'(x))^2\,dx\geqslant\frac{n^2}{2(2n-1)}P_n^2(1).$$

The author is grateful to Professor P. Erdös for some valuable conversation.

REFERENCES

1. J. Eröd, *Bizonyos polinomok maximumar ial*, Mat. Fiz. Lapok **46** (1939), 58.

2. J. Szabadös and A. K. Varma, *Approximation theory*. III (E. W. Cheney, ed.), Academic Press, New York, 1980, pp. 881–888.

3. P. Turán, *Über die Ableitund von Polynomen*, Compositio Math. **7** (1939), 89–95.

4. A. K. Varma, *An analogue of some inequalities of P. Turán concerning algebraic polynomials having all zeros inside* [−1, 1], Proc. Amer. Math. Soc. **69** (1978), 25–33.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF FLORIDA, GAINESVILLE, FLORIDA 32601