SOME INEQUALITIES OF ALGEBRAIC POLYNOMIALS HAVING ALL ZEROS INSIDE \([-1,1]\)

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Abstract. Let \( H_n \) be the set of all algebraic polynomials whose degree is \( n \) and whose zero are all real and lie inside \([-1,1]\). Then for \( n \) even we have (\( n = 2m \))

\[
\int_{-1}^{1} (P_n(x))^2 \, dx \geq \left( \frac{n}{2} + \frac{3}{4} + \frac{3}{4}(n-1) \right) \int_{-1}^{1} P_n^2(x) \, dx,
\]

where equality holds iff \( P_n(x) = (1 - x^2)^m \). If \( n \) is an odd positive integer, a similar inequality is valid (see (1.6) below). In the case \( P_n \in H_n \) and subject to the condition \( P_n(1) = 1 \), then

\[
\int_{-1}^{1} (P_n'(x))^2 \, dx \geq \frac{n}{4} + \frac{1}{8} + \frac{1}{8(2n-1)},
\]

where equality holds for \( P_n(x) = ((1 + x)/2)^m \).

Let \( H_n \) be the set of all polynomials whose degree is \( n \) and whose zeros are all real and lie inside \([-1,1]\). In this work we are concerned with the following theorems of P. Turán [3], A. K. Varma [4], and J. Szabados and A. K. Varma [2].

Theorem A (P. Turán). Let \( P_n \in H_n \); then

\[
\max_{-1 \leq x \leq 1} \left| P_n'(x) \right| \geq \left( \frac{n}{6} \right)^{1/2} \max_{-1 \leq x \leq 1} \left| P_n(x) \right|.
\]

(\( P_n'(x) \) stands for the derivative of \( P_n(x) \).)

Theorem B (A. K. Varma). Let \( P_n \in H_n \); then

\[
\| P_n' \|_{L_2[-1,1]} \geq \left( \frac{n}{2} + \frac{3}{4} + \frac{3}{4n} \right) \| P_n \|_{L_2[-1,1]}, \quad n \geq 13,
\]

where

\[
\| P_n \|_{L_2[-1,1]} = \int_{-1}^{1} P_n^2(x) \, dx.
\]

Further, if \( P_n(1) = P_n(-1) = 0 \) and \( n \geq 2 \), \( P_n \in H_n \), then

\[
\| P_n' \|_{L_2[-1,1]} \geq \left( \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)} \right) \| P_n \|_{L_2[-1,1]},
\]

where equality holds for \( P_n(x) = (1 - x^2)^m \), \( n = 2m \).
THEOREM C (J. Szabadös and A. K. Varma). Let $P_n \in H_n$, subject to the condition $P_n(1) = 1$; then

$$\|P_n\|^2_{L_2[-1,1]} > n/4.$$  

The object of this paper is to prove the following theorems:

**Theorem 1.** Let $P_n \in H_n$ and $n = 2m$; then

$$\|P_n\|^2_{L_2[-1,1]} \geq \left( \frac{n}{2} + \frac{3}{4} + \frac{3}{4(n-1)} \right) \|P_n\|^2_{L_2[-1,1]},$$

where equality holds iff $P_n(x)(1 - x^2)^m$. Moreover, if $n = 2m - 1$, then

$$\|P_n\|^2_{L_2[-1,1]} \geq \left( \frac{n}{2} + \frac{3}{4} + \frac{5}{4(n-2)} \right) \|P_n\|^2_{L_2[-1,1]}, \quad n \geq 3,$$

where equality holds iff

$$P_n(x) = (1 - x)^{m-1}(1 + x)^m \quad \text{or} \quad P_n(x) = (1 - x)^m(1 + x)^{m-1}.$$  

**Theorem 2.** Let $P_n \in H_n$, subject to the condition $P_n(1) = 1$; then

$$\|P_n\|^2_{L_2[-1,1]} \geq \frac{n}{4} + \frac{1}{8} + \frac{1}{8(2n-1)}, \quad n \geq 1,$$

where equality holds for $P_n(x) = ((1 + x)/2)^n$.

**Remark.** Theorem 1 is an improvement over Theorem B in two respects. First, the condition $P_n(1) = P_n(-1) = 0$ is not necessary for (1.3) to hold. Secondly, we have precise bounds for $n$ even and for $n$ odd as mentioned in (1.5) and (1.6). Also note that (1.7) is an improvement over (1.4). A more precise form of Theorem A is due to J. Erőd [1].

**2. Preliminaries.** The following results are known:

(2.1)  
$$P_n(x) = P_n(x) \sum_{k=1}^{n} \frac{1}{x - x_k},$$

(2.2)  
$$(P_n'(x))^2 - P_n(x)P_n''(x) = (P_n(x))^2 \sum_{k=1}^{n} \frac{1}{(x - x_k)^2},$$

where $x_1, x_2, \ldots, x_n$ denote the roots of $P_n(x)$ such that

$$-1 \leq x_n \leq x_{n-1} \leq \cdots \leq x_2 \leq x_1 \leq 1.$$  

We also note that

$$\frac{(x_1 - x)(x - x_n)}{(x - x_k)^2} = 1 - \frac{2x}{x - x_k} + \frac{x_1 + x_n}{x - x_k} + \frac{(x_1 - x_k)(x_k - x_n)}{(x - x_k)^2}. $$

**3. Some identities.** For the proof of Theorem 1 the following identities are needed.  

**Identity 3.1.** Let

(3.1)  
$$\alpha(x) = (x_1 - x)(x - x_n),$$
and let \( P_n(x) \) be any polynomial of degree \( n \) having \( x_1, x_2, \ldots, x_n \) as its roots. Then

\[
2 \int_{x_n}^{x_1} (P'_n(x))^2 \, dx = (n + 1) \int_{x_n}^{x_1} \frac{P^2_n(x)}{\alpha(x)} \, dx + \lambda_0 + \frac{1}{2} \int_{x_n}^{x_1} P^2_n(x) (\alpha'(x))^2 \, dx,
\]

where

\[
\lambda_0 = \int_{x_n}^{x_1} \frac{P^2_n(x)}{\alpha(x)} \sum_{k=1}^{n} \frac{(x_1 - x_k)(x_k - x_n)}{(x - x_k)^2} \, dx.
\]

**Proof.** On integrating by parts and using \( P_n(x_1) = P_n(x_n) = 0 \), we obtain

\[
2 \int_{x_n}^{x_1} (P'_n(x))^2 \, dx = \int_{x_n}^{x_1} \left( (P'_n(x))^2 - P_n(x)P''_n(x) \right) \, dx.
\]

Next, on using (2.2) and (2.4), we obtain

\[
2 \int_{x_n}^{x_1} (P'_n(x))^2 \, dx = n \int_{x_n}^{x_1} \frac{P^2_n(x)}{\alpha(x)} \, dx + \lambda_0 + \int_{x_n}^{x_1} \frac{\alpha'(x)}{\alpha(x)} P_n(x)P'_n(x) \, dx.
\]

But integrating by parts gives us

\[
\int_{x_n}^{x_1} \frac{\alpha'(x)}{\alpha(x)} P_n(x)P'_n(x) \, dx = \int_{x_n}^{x_1} \frac{P^2_n(x)}{\alpha(x)} \, dx + \frac{1}{2} \int_{x_n}^{x_1} \frac{P^2_n(x)(\alpha'(x))^2}{\alpha^2(x)} \, dx.
\]

From (3.4) and (3.5), (3.2) follows.

Next, we shall prove

**Identity 3.2.** Let \( P_n(x) \) be any polynomial of degree \( n \) having \( x_1, x_2, \ldots, x_n \) as its roots; then

\[
2 \int_{x_n}^{x_1} (P'_n(x))^2 \alpha(x) \, dx = n \int_{x_n}^{x_1} P^2_n(x) \, dx + S_0,
\]

where

\[
S_0 = \int_{x_n}^{x_1} \sum_{k=1}^{n} \frac{(x_1 - x_k)(x_k - x_n)}{(x - x_k)^2} P^2_n(x) \, dx,
\]

and \( \alpha(x) \) is as defined by (3.1).

**Proof.** By integrating by parts, we have

\[
2 \int_{x_n}^{x_1} (P'_n(x))^2 \alpha(x) \, dx = \int_{x_n}^{x_1} \left( (P'_n(x))^2 - P_n(x)P''_n(x) \right) \alpha(x) \, dx
\]

\[
- \int_{x_n}^{x_1} P_n(x)P'_n(x) \alpha'(x) \, dx
\]

\[
= \int_{x_n}^{x_1} P^2_n(x) \sum_{k=1}^{n} \frac{\alpha(x)}{(x - x_k)^2} \, dx - \int_{x_n}^{x_1} P^2_n(x) \sum_{k=1}^{n} \frac{\alpha'(x)}{x - x_k} \, dx
\]

\[
= \int_{x_n}^{x_1} P^2_n(x) \sum_{k=1}^{n} \left( \frac{\alpha(x)}{(x - x_k)^2} - \frac{\alpha'(x)}{x - x_k} \right) \, dx.
\]

By using (2.4) and (3.7) we obtain (3.6).
Now, we turn to prove

**Identity 3.3.** Let \( P_n(x) \) be any polynomial of degree \( n \) having \( x_1, x_2, \ldots, x_n \) as its zeros; then (for \( n = 2m \))

\[
(3.9) \quad I_n \equiv \int_{x_n}^{x_1} \left( \sqrt{\alpha(x)} P_n'(x) - \frac{m\alpha'(x)}{\sqrt{\alpha(x)}} P_n(x) \right)^2 \, dx
\]

\[
= -\frac{n}{2} (2n + 1) \int_{x_n}^{x_1} P_n^2(x) \, dx + \frac{1}{2} S_0 + (x_1 - x_n)^2 m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx,
\]

where \( S_0 \) is defined by (3.7). Further,

\[
(3.10) \quad I'_n \equiv \int_{x_n}^{x_1} \left( P_n'(x) - \frac{m\alpha'(x)}{\alpha(x)} P_n(x) \right)^2 \, dx
\]

\[
= \int_{x_n}^{x_1} (P_n'(x))^2 \, dx + m(m - 1) \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)} \, dx - n \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx.
\]

**Proof.** From the definition of \( I_n \) and using identity (3.2), we have

\[
I_n = \frac{n}{2} \int_{x_n}^{x_1} P_n^2(x) \, dx + \frac{1}{2} S_0 - n \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx + m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha(x)} \, dx.
\]

But

\[
(\alpha'(x))^2 = (x_1 - x_n)^2 - 4\alpha(x).
\]

Therefore

\[
I_n = -\frac{n}{2} \int_{x_n}^{x_1} P_n^2(x) \, dx + \frac{1}{2} S_0 - n^2 \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx + (x_1 - x_n)^2 m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx.
\]

this proves (3.9). Similarly from the definition of \( I'_n \) as given in (3.10), we have

\[
(3.11) \quad I'_n = \int_{x_n}^{x_1} (P_n'(x))^2 \, dx + m^2 \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)} \, dx
\]

\[
- n \int_{x_n}^{x_1} \frac{P_n(x)P_n'(x)}{\alpha(x)} \frac{\alpha'(x)}{\alpha(x)} \, dx.
\]

But

\[
(3.12) \quad \int_{x_n}^{x_1} P_n(x)P_n'(x) \frac{\alpha'(x)}{\alpha(x)} \, dx = -\frac{1}{2} \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha''(x)\alpha(x) - (\alpha'(x))^2)}{\alpha^2(x)} \, dx
\]

\[
= \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx + \frac{1}{2} \int_{x_n}^{x_1} \frac{P_n^2(x)(\alpha'(x))^2}{\alpha^2(x)} \, dx.
\]

From (3.11) and (3.12), (3.10) follows at once.
4. Proof of Theorem 1. Let \( P_n \in H_n, P_n(x_1) = P_n(x_n) = 0 \). Since \( P_n(x) \) has no zeros inside \([x_1, 1]\),
\[
\frac{1}{n} \sum_{k=1}^{n} \frac{1}{x - x_k} - \frac{n}{2} = \frac{1}{n} \sum_{k=1}^{n} \left( \frac{1}{x - x_k} - \frac{1}{2} \right) = \sum_{k=1}^{n} \frac{2 - (x - x_k)}{2(x - x_k)} > 0
\]
for \( x_1 \leq x \leq 1 \). Therefore, using (2.1),
\[
(4.1) \quad \int_{x_1}^{1} (P_n^2(x))^2 \, dx = \int_{x_1}^{1} P_n^2(x) \left( \sum_{k=1}^{n} \frac{1}{x - x_k} \right)^2 \, dx \geq \frac{n^2}{4} \int_{x_1}^{1} P_n^2(x) \, dx.
\]
Similarly,
\[
(4.2) \quad \int_{-1}^{x_1} (P_n^2(x))^2 \, dx \geq \frac{n^2}{4} \int_{-1}^{x_1} P_n^2(x) \, dx.
\]
Next we consider two cases. First let \( x_1 = x_n = x_k, k = 1, \ldots, n \). Then in view of (4.1) and (4.2),
\[
(4.3) \quad \int_{x_1}^{1} (P_n^2(x))^2 \, dx \geq \frac{n^2}{4} \int_{-1}^{x_1} P_n^2(x) \, dx
\]
So we turn to the case when \( x_1 \neq x_n \). On using (3.2) and (3.10) we obtain \( (n = 2m, m > 1) \)
\[
2 \int_{x_n}^{x_1} (P_n'(x))^2 \, dx = \lambda_0 + (n + 1) \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx
\]
\[
+ \frac{1}{2m(m - 1)} \left[ I_n' + n \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx - \int_{x_n}^{x_1} (P_n'(x))^2 \, dx \right].
\]
Since \( I_n' \geq 0 \) and equal to zero iff \( P_n(x) = (1 - x^2)^m, n = 2m \), we can rewrite the above as
\[
\frac{(n - 1)^2}{n(m - 1)} \int_{x_n}^{x_1} (P_n'(x))^2 \, dx \geq \frac{m(n - 1)}{(m - 1)} \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx + \lambda_0.
\]
Next, using (3.9) and observing that \( I_n \geq 0 = 0 \) iff \( P_n(x) = (1 - x^2)^m, n = 2m \), we obtain
\[
(4.4) \quad \frac{(n - 1)^2}{n(m - 1)} \int_{x_n}^{x_1} (P_n'(x))^2 \, dx
\]
\[
\geq \lambda_0 + \frac{(n - 1)}{m(m - 1)(x_1 - x_n)} \left[ -\frac{1}{2} S_0 + \frac{n(2n + 1)}{2} \int_{x_n}^{x_1} \frac{P_n^2(x)}{\alpha(x)} \, dx \right].
\]
But using (3.3), (3.7) and
\[
\alpha(x) = (x_1 - x)(x - x_n) \leq (x_1 - x_n)^2, \quad x_1 \leq x \leq x_1,
\]
we obtain

$$\lambda_0 \geq (x_1 - x_n)^2 S_0.$$ 

Therefore (4.4) becomes

$$\int_{x_n}^{x_1} (P_n'(x))^2 \, dx \geq \frac{(n - 1)^2}{n(m - 1)} \int_{x_n}^{x_1} \left( P_n'(x) \right)^2 \, dx \geq \frac{2n^2 - 2n + 1}{n(m - 1)} S_0 + \frac{(n - 1)(2n + 1)}{(m - 1)(x_1 - x_n)^2} \int_{x_n}^{x_1} P_n^2(x) \, dx.$$ 

Since $$S_0 > 0$$, $$m > 1$$, we have

$$\int_{x_n}^{x_1} (P_n'(x))^2 \, dx \geq \frac{(2n + 1)n}{(n - 1)(x_1 - x_n)^2} \int_{x_n}^{x_1} P_n^2(x) \, dx. \tag{4.5}$$

On using (4.1), (4.2) and (4.5) we obtain

$$\int_{-1}^{1} (P_n'(x))^2 \, dx \geq \frac{n^2}{4} \int_{-1}^{1} P_n^2(x) \, dx + \frac{n^2}{4} \int_{x_1}^{1} P_n^2(x) \, dx + \frac{(2n + 1)n}{(n - 1)(x_1 - x_n)^2} \int_{x_n}^{x_1} P_n^2(x) \, dx$$

$$\geq \frac{(2n + 1)n}{4(n - 1)} \int_{-1}^{1} P_n^2(x) \, dx \tag{for \, n \geq 4}$$ 

where equality holds when $$x_1 = 1, x_n = -1$$. This proves Theorem 1 for n even. For n odd the proof of (1.6) is similar, so we omit the details. Further, for $$n = 1, 2$$, inequality (1.5) and (1.6) can be easily verified.

**Proof of Theorem 2.** Let $$P_n \in H_n, P_n(1) = 1$$, and let the zeros of $$P_n(x)$$ be given by

$$-1 \leq x_n \leq x_{n-1} \leq \cdots \leq x_2 \leq x_1 < 1. \tag{4.4}$$

Since

$$\int_{x_1}^{1} (P_n'(x))^2 \, dx = \int_{x_1}^{1} \left( P_n'(x) - \frac{n P_n'(x)}{x - x_n} \right)^2 \, dx$$

$$-n^2 \int_{x_1}^{1} \frac{P_n^2(x)}{(x - x_n)^2} \, dx + 2n \int_{x_1}^{1} \frac{P_n(x) P_n'(x)}{x - x_n} \, dx.$$ 

But, on integrating by parts,

$$\int_{x_1}^{1} \frac{2P_n(x) P_n'(x)}{x - x_n} \, dx = \frac{P_n^2(1)}{1 - x_n} + \int_{x_1}^{1} \frac{P_n^2(x)}{(x - x_n)^2} \, dx.$$ 

$$\int_{x_1}^{1} \frac{2P_n(x) P_n'(x)}{x - x_n} \, dx = \frac{P_n^2(1)}{1 - x_n} + \int_{x_1}^{1} \frac{P_n^2(x)}{(x - x_n)^2} \, dx.$$
Using (4.5) and (4.6) we obtain

\[
\int_{x_1}^{1} (P_n'(x))^2 \, dx = \int_{x_1}^{1} \left( P_n'(x) - \frac{n P_n(x)}{x - x_n} \right)^2 \, dx + \frac{n P_n(1)}{1 - x_n} - n(n - 1) \int_{x_1}^{1} \frac{P_n^2(x)}{(x - x_n)^2} \, dx.
\]

Next we note that

\[
\int_{x_1}^{1} (P_n'(x))^2 \, dx = \int_{x_1}^{1} P_n^2(x) \left( \sum_{k=1}^{n} \frac{1}{x - x_k} \right)^2 \, dx.
\]

Further

\[
x - x_k \leq x - x_n, \quad k = 1, 2, \ldots, n; \ x_1 \leq x \leq 1.
\]

Therefore

\[
\int_{x_1}^{1} (P_n'(x))^2 \, dx \geq n^2 \int_{x_1}^{1} \frac{P_n^2(x)}{(x - x_n)^2} \, dx.
\]

Thus, from (4.7) and (4.8),

\[
\int_{x_1}^{1} (P_n'(x))^2 \, dx \geq \int_{x_1}^{1} \left( P_n'(x) - \frac{n P_n(x)}{x - x_n} \right)^2 \, dx + \frac{n P_n^2(1)}{1 - x_n} - \frac{(n - 1)}{n} \int_{x_1}^{1} (P_n'(x))^2 \, dx.
\]

In other words,

\[
\frac{(2n - 1)}{n} \int_{x_1}^{1} (P_n'(x))^2 \, dx \geq \int_{x_1}^{1} \left( P_n'(x) - \frac{n P_n(x)}{x - x_n} \right)^2 \, dx + \frac{n P_n^2(1)}{1 - x_n} + \int_{x_1}^{1} \left( P_n'(x) - \frac{n P_n(x)}{x - x_n} \right)^2 \, dx.
\]

From this we conclude that

\[
\int_{x_1}^{1} (P_n'(x))^2 \, dx \geq \frac{n^2}{(2n - 1)(1 - x_n)} P_n^2(1) \geq \frac{n^2}{2(2n - 1)} P_n^2(1)
\]

where equality holds iff \( x_n = -1 \) and \( P_n'(x) - n P_n(x)/(x + 1) = 0 \). But then \( P_n(x) = (1 + x)^n/2^n \) and all roots of \( P_n(x) \) are equal to \(-1\). From this we finally have

\[
\int_{-1}^{1} (P_n'(x))^2 \, dx \geq \frac{n^2}{2(2n - 1)} P_n^2(1).
\]

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References


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