PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 2, June 1983

INTERPOLATION BETWEEN H^1 AND L^∞

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ABSTRACT. The intermediate spaces in the Lions-Peetre method of interpolation between H^1 and L^∞ were identified by N. Rivière and Y. Sagher as Lorentz $L(p,q)$ spaces. In this article we present a simplification of their proof of this result.

1. Introduction. The real interpolation spaces between H^1 and L^∞ were first identified as the Lorentz $L(p,q)$ spaces by N. Rivière and Y. Sagher [**5**]. A proof of this result by quite different methods has been recently given by C. Bennett and R. Sharpley [**2**, pp. 111–116]. The purpose of this article is to give a simplification of the original proof given in [**5**]. In particular, the details of the computation of the main norm estimate will be simpler, and the variation of Hardy's inequality used by Rivière and Sagher will no longer be necessary.

We begin with a brief review of some definitions and notation. The (real) Hardy space $H^1(\mathbf{R}^n)$ is the space of functions f in L^1 for which $\|f\|_{H^1} = \|f\|_{L^1} + \sum_{j=1}^n \|R_j f\|_{L^1}$ is finite, where R_j is the Riesz transform

$$R_j f(x) = \text{p.v.}\int_{\mathbf{R}^n} \frac{f(x-y)y_j}{|y|^{n+1}}\,dy.$$

The Lorentz space $L(p,q)$ is the collection of all measurable complex-valued functions f such that $\|f\|_{p,q}$ is finite, where

$$\|f\|_{p,q} = \begin{cases} \left(\dfrac{q}{p}\displaystyle\int_0^\infty \left[t^{1/p}f^*(t)\right]^q dt/t\right)^{1/q}, & 0 < p, q < \infty, \\ \displaystyle\sup_{t>0} t^{1/p}f^*(t), & q = \infty, 0 < p \leqslant \infty. \end{cases}$$

Recall that f^* is the nonincreasing rearrangement of f; a nonnegative, right continuous function on $(0,\infty)$ which is equimeasurable with f.

The K-functional of a compatible Banach couple (A_0, A_1) is defined for f in $A_0 + A_1$ and $t > 0$ by

$$K(t, f; A_0, A_1) = \inf\left\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : f = f_0 + f_1, f_0 \in A_0 \text{ and } f_1 \in A_1\right\}.$$

This will be abbreviated as $K(t,f)$ when no confusion can result. For $0 < \theta < 1$ and $1 \leqslant q \leqslant \infty$, the real interpolation space $(A_0, A_1)_{\theta,q}$ consists of all f in $A_0 + A_1$ for

Received by the editors September 30, 1981.
1980 *Mathematics Subject Classification*. Primary 46E30; Secondary 46M35, 42B30.
Key words and phrases. Real interpolation, H^1, Lorentz spaces.

which

$$\|f\|_{(A_0,A_1)_{\theta,q}} \equiv \|f\|_{\theta,q} = \begin{cases} \left(\int_0^\infty \left[t^{-\theta}K(t,f)\right]^q dt/t\right)^{1/q} & (q<\infty), \\ \sup_{t>0} t^{-\theta}K(t,f) & (q=\infty) \end{cases}$$

if finite.

2. Interpolation between H^1 and L^∞. We now state the theorem of Rivière and Sagher and give our simplification of the original proof.

THEOREM [5, §2, THEOREM 4]. $(H^1, L^\infty)_{\theta,q} = L(p,q)$ *where* $1/p = 1-\theta$, $1 \leq q \leq \infty$.

PROOF. Since $(H^1, L^\infty)_{\theta,q} \subseteq (L^1, L^\infty)_{\theta,q} = L(p,q)$ [**1**, p. 109], we need only show the reverse inclusion. Choose f in $L(p,q)$. We will show that $\|f\|_{\theta,q} \leq C\|f\|_{p,q}$ where C is a constant independent of f.

Fix a positive number t and an r such that $1<r<p$. Define the maximal function

$$M_rf(x) = \sup_{x\in I}\left(\frac{1}{|I|}\int_I |f|^r\right)^{1/r}, \tag{1}$$

where I denotes a cube in $\mathbf{R}^n$. Set $\alpha = (M_rf)^*(t)$.

The Calderon-Zygmund decomposition of $\mathbf{R}^n$ (relative to the function $|f|^r$ and constant α^r) gives a collection of cubes $\{I_j\}$ with disjoint interiors so that

$$\alpha < \left(\frac{1}{|I_j|}\int_{I_j}|f|^r\right)^{1/r} \leq 2^n\alpha \tag{2}$$

and

$$|f(x)| \leq \alpha \quad \text{for a.e. } x \notin \bigcup I_j. \tag{3}$$

Now set

$$f' = \sum_j (f - f_{I_j})\chi_{I_j} \quad \text{where } f_{I_j} = \frac{1}{|I_j|}\int_{I_j} f. \tag{4}$$

This gives the decomposition $f = f' + (f - f')$. We remark that Rivière and Sagher obtain a decomposition of f in the same manner, but use the more complicated value $(\frac{1}{t}\int_0^t[(M_rf)^*(s)]^r\,ds)^{1/r}$ for α.

We next obtain upper estimates for $\|f'\|_{H^1}$ and $\|f-f'\|_\infty$. Note that

$$f - f' = \begin{cases} f & \text{on } \mathbf{R}^n \setminus \bigcup I_j, \\ f_{I_j} & \text{on } I_j. \end{cases}$$

An estimate using Holder's inequality and (2) shows $|f_{I_j}| \leq 2^n\alpha$. Combining this with (3) gives $\|f-f'\|_\infty \leq 2^n\alpha$.

To obtain an estimate on $\|f'\|_{H^1}$, note that $(f-f_{I_j})\chi_{I_j}$ is supported in I_j and has average 0 over it. Thus, by Lemma 2 of [**5**],

$$\|(f-f_{I_j})\chi_{I_j}\|_{H^1} \leq C|I_j|^{1-1/r}\|(f-f_{I_j})\chi_{I_j}\|_{L^r}. \tag{5}$$

Minkowski's inequality and the above estimate on $|f_{I_j}|$ together with (2) give

(6) $$\|(f-f_{I_j})\chi_{I_j}\|_{L^r} \leqslant 2^n\alpha\,|I_j|^{1/r}.$$

From (4)–(6) we have

(7) $$\|f^t\|_{H^1} \leqslant C\alpha\sum_j |I_j|.$$

The cubes I_j have disjoint interiors so $\Sigma_j|I_j| = |\cup I_j|$. If x is in some I_j, then by (1) and (2), $M_rf(x) > \alpha$. Thus $\cup I_j \subseteq \{x\colon\ M_rf(x) > \alpha\} = \{x\colon\ M_rf(x) > (M_rf)^*(t)\}$. Moreover, since M_rf and $(M_rf)^*$ have the same distribution function,

$$|\{x\colon M_rf(x) > (M_rf)^*(t)\}| = |\{s\colon (M_rf)^*(s) > (M_rf)^*(t)\}| \leqslant t.$$

Thus $|\cup I_j| \leqslant t$. By (7) this gives $\|f^t\|_{H^1} \leqslant C\alpha t$.

The estimates on $\|f^t\|_{H^1}$ and $\|f-f^t\|_\infty$ give

$$K(t,f;H^1,L^\infty) \leqslant \|f^t\|_{H^1} + t\|f-f^t\|_\infty \leqslant C\alpha t = Ct(M_rf)^*(t).$$

Therefore, for $q<\infty$,

$$\begin{aligned}\|f\|_{\theta,q} &= \left(\int_0^\infty\left[t^{-\theta}K(t,f)\right]^q dt/t\right)^{1/q}\\ &\leqslant C\left(\int_0^\infty\left[t^{1-\theta}(M_rf)^*(t)\right]^q dt/t\right)^{1/q}\\ &= C\|M_rf\|_{p,q} \quad \text{where } 1/p = 1-\theta\\ &\leqslant C\|f\|_{p,q}.\end{aligned}$$

The last inequality follows from the fact that M_r is a bounded sublinear operator from $L(r,r)$ to $L(r,\infty) = $ weak L^r and also from L^∞ to L^∞ and, hence [**4**, p. 264], M_r is bounded from $L(p,q)$ to $L(p,q)$ where $p>r$. The argument for $q=\infty$ is entirely similar to the above.

We remark that $\|f^t\|_{H^1}$ could also be estimated by noting that

$$a_j = (f-f_{I_j})\chi_{I_j}\Big/\left(2^n\alpha\,|I_j|\right)$$

is a $(1,r)$ atom in the sense of [**3**, p. 591]. Thus $f^t = \Sigma_j 2^n\alpha\,|I_j|\,a_j$ has H^1-norm $\leqslant C\Sigma\alpha\,|I_j|$ as before.

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