INTERPOLATION BETWEEN $H^1$ AND $L^\infty$

BARBARA D. MACCLUER

Abstract. The intermediate spaces in the Lions-Peetre method of interpolation between $H^1$ and $L^\infty$ were identified by N. Rivière and Y. Sagher as Lorentz $L(p, q)$ spaces. In this article we present a simplification of their proof of this result.

1. Introduction. The real interpolation spaces between $H^1$ and $L^\infty$ were first identified as the Lorentz $L(p, q)$ spaces by N. Rivière and Y. Sagher [5]. A proof of this result by quite different methods has been recently given by C. Bennett and R. Sharpley [2, pp. 111–116]. The purpose of this article is to give a simplification of the original proof given in [5]. In particular, the details of the computation of the main norm estimate will be simpler, and the variation of Hardy’s inequality used by Rivière and Sagher will no longer be necessary.

We begin with a brief review of some definitions and notation. The (real) Hardy space $H^1(\mathbb{R}^n)$ is the space of functions $f$ in $L^1$ for which $\| f \|_{H^1} = \| f \|_{L^1} + \sum_{j=1}^\infty \| R_j f \|_L$ is finite, where $R_j$ is the Riesz transform

$$R_j f(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{f(x-y)y_j}{|y|^{n+1}} \, dy.$$ 

The Lorentz space $L(p, q)$ is the collection of all measurable complex-valued functions $f$ such that $\| f \|_{L(p, q)}$ is finite, where

$$\| f \|_{L(p, q)} = \begin{cases} \left( \frac{q}{p} \int_0^\infty \left( \frac{t^{1/p} f^*(t)}{t} \right)^q \, dt/t \right)^{1/q}, & 0 < p, q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & q = \infty, 0 < p \leq \infty. \end{cases}$$

Recall that $f^*$ is the nonincreasing rearrangement of $f$; a nonnegative, right continuous function on $(0, \infty)$ which is equimeasurable with $f$.

The $K$-functional of a compatible Banach couple $(A_0, A_1)$ is defined for $f$ in $A_0 + A_1$ and $t > 0$ by

$$K(t, f; A_0, A_1) = \inf \{ \| f_0 \|_{A_0} + t \| f_1 \|_{A_1} : f = f_0 + f_1, f_0 \in A_0 \text{ and } f_1 \in A_1 \}.$$ 

This will be abbreviated as $K(t, f)$ when no confusion can result. For $0 < \theta < 1$ and $1 \leq q \leq \infty$, the real interpolation space $(A_0, A_1)_{\theta, q}$ consists of all $f$ in $A_0 + A_1$ for
which

\[ \| f \|_{(A_0,A_1)_{\theta,q}} \equiv \| f \|_{\theta,q} = \begin{cases} \left( \int_0^\infty \left[ \frac{t^{-\theta}K(t,f)}{dt/t} \right]^q dt/t \right)^{1/q} & (q < \infty), \\ \sup_{t>0} t^{-\theta}K(t,f) & (q = \infty) \end{cases} \]

if finite.

2. Interpolation between \( H^1 \) and \( L^\infty \). We now state the theorem of Rivière and Sagher and give our simplification of the original proof. 

**Theorem [5, §2, Theorem 4].** \( (H^1, L^\infty)_{\theta,q} = L(p,q) \) where \( 1/p = 1 - \theta, 1 \leq q \leq \infty \).

**Proof.** Since \( (H^1, L^\infty)_{\theta,q} \subseteq (L^1, L^\infty)_{\theta,q} = L(p,q) [1, p. 109] \), we need only show the reverse inclusion. Choose \( f \) in \( L(p,q) \). We will show that \( \| f \|_{\theta,q} \leq C \| f \|_{p,q} \) where \( C \) is a constant independent of \( f \).

Fix a positive number \( t \) and an \( r \) such that \( 1 < r < p \). Define the maximal function

\[ M_t f(x) = \sup_{x \in I} \left\{ \frac{1}{|I|} \int_I |f(x')|^{r'} \right\}^{1/r}, \]

where \( I \) denotes a cube in \( \mathbb{R}^n \). Set \( \alpha = (M_t f)^*(t) \).

The Calderon-Zygmund decomposition of \( \mathbb{R}^n \) (relative to the function \( |f(x')|^{r'} \) and constant \( \alpha^r \)) gives a collection of cubes \( \{I_j\} \) with disjoint interiors so that

\[ \alpha < \left( \frac{1}{|I_j|} \int_{I_j} |f(x')|^{r'} \right)^{1/r} \leq 2^n \alpha \]

and

\[ |f(x)| \leq \alpha \text{ for a.e. } x \notin \bigcup I_j. \]

Now set

\[ f' = \sum_j (f - f_{I_j})\chi_{I_j} \text{ where } f_{I_j} = \frac{1}{|I_j|} \int_{I_j} f. \]

This gives the decomposition \( f = f' + (f - f') \). We remark that Rivière and Sagher obtain a decomposition of \( f \) in the same manner, but use the more complicated value \((\frac{1}{|I_j|} \int (M_t f)^*(s))^{1/r} ds)^{1/r} \) for \( \alpha \).

We next obtain upper estimates for \( \| f' \|_{H^1} \) and \( \| f - f' \|_{\infty} \). Note that

\[ f - f' = \begin{cases} f & \text{on } \mathbb{R}^n \setminus \bigcup I_j, \\ f_{I_j} & \text{on } I_j. \end{cases} \]

An estimate using Hölder’s inequality and (2) shows \( |f_{I_j}| \leq 2^n \alpha \). Combining this with (3) gives \( \| f - f' \|_{\infty} \leq 2^n \alpha \).

To obtain an estimate on \( \| f' \|_{H^1} \), note that \( (f - f_{I_j})\chi_{I_j} \) is supported in \( I_j \) and has average 0 over it. Thus, by Lemma 2 of [5],

\[ \| (f - f_{I_j})\chi_{I_j} \|_{H^1} \leq C |I_j|^{-1/r}(f - f_{I_j})\chi_{I_j} \|_{L^r}. \]
Minkowski’s inequality and the above estimate on $|f_j|$ together with (2) give
\begin{equation}
\| (f - f_j)x_j \|_{L'} \leq 2^n \alpha |I_j|^{1/r}.
\end{equation}

From (4)–(6) we have
\begin{equation}
\| f' \|_{H^1} \leq C\alpha \sum_j |I_j|.
\end{equation}

The cubes $I_j$ have disjoint interiors so $\sum_j |I_j| = | \bigcup I_j |$. If $x$ is in some $I_j$, then by (1) and (2), $M_r f(x) > \alpha$. Thus $\bigcup I_j \subseteq \{ x : M_r f(x) > \alpha \} = \{ x : M_r f(x) > (M_r f)^*(t) \}$. Moreover, since $M_r f$ and $(M_r f)^*$ have the same distribution function,
\begin{equation*}
| \{ x : M_r f(x) > (M_r f)^*(t) \} | = | \{ s : (M_r f)^*(s) > (M_r f)^*(t) \} | \leq t.
\end{equation*}

Thus $| \bigcup I_j | \leq t$. By (7) this gives $\| f' \|_{H^1} \leq C\alpha t$.

The estimates on $\| f' \|_{H^1}$ and $\| f - f' \|_{\infty}$ give
\begin{equation*}
K(t, f; H^1, L^\infty) \leq \| f' \|_{H^1} + t \| f - f' \|_{\infty} \leq C\alpha t = C(t (M_r f)^*(t)).
\end{equation*}

Therefore, for $q < \infty$,
\begin{equation*}
\| f \|_{\theta, q} = \left( \int_0^\infty \left[ t^{-\theta} K(t, f) \right]^q \frac{dt}{t} \right)^{1/q} \\
\leq C \left( \int_0^\infty \left[ t^{1-\theta} (M_r f)^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \\
= C \| M_r f \|_{p, q} \quad \text{where } 1/p = 1 - \theta \\
\leq C \| f \|_{p, q}.
\end{equation*}

The last inequality follows from the fact that $M_r$ is a bounded sublinear operator from $L(r, r)$ to $L(r, \infty) = \text{weak } L'$ and also from $L^\infty$ to $L^\infty$ and, hence [4, p. 264], $M_r$ is bounded from $L(p, q)$ to $L(p, q)$ where $p > r$. The argument for $q = \infty$ is entirely similar to the above.

We remark that $\| f' \|_{H^1}$ could also be estimated by noting that
\begin{equation*}
a_j = (f - f_j)x_j \bigg/ (2^n \alpha |I_j|)
\end{equation*}
is a $(1, r)$ atom in the sense of [3, p. 591]. Thus $f' = \sum_j 2^n \alpha |I_j| a_j$ has $H^1$-norm $\leq C\alpha \sum |I_j|$ as before.

REFERENCES


Department of Mathematics, Michigan State University, East Lansing, Michigan 48824