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ON FOURIER TRANSFORMS OF DISTRIBUTIONS WITH ONE-SIDED BOUNDED SUPPORT

R. SHAMBAYATI AND Z. ZIELEZNY

ABSTRACT. Fourier transforms of distributions of finite order with left-sided bounded support are characterized. Furthermore, a product of these Fourier transforms is defined which corresponds to the convolution of the original distributions.

Let $\mathcal{D}'$ be the space of distributions on $\mathbf{R}$, and let $\mathcal{D}'_+$ be the subspace of $\mathcal{D}'$ consisting of distributions with left-sided bounded support (see [**5**]). Furthermore, let D' be the space of Fourier transforms of distributions in $\mathcal{D}'$ as defined in [**2** and **3**]. It seems natural to pose the following problems:

(I) Characterize the subspace D'_+ of D' consisting of Fourier transforms of distributions in $\mathcal{D}'_+$.

(II) Define in D'_+ a product which would correspond under the Fourier transform to the convolution. More precisely, if S and T are distributions in $\mathcal{D}'_+$ and $S * T$ is their convolution, define a product $\hat{S} \circ \hat{T}$ of the Fourier transforms of S and T so that $(S * T)^{\hat{}} = \hat{S} \circ \hat{T}$.

In this paper we solve problems (I) and (II) for the space $\mathcal{S}'_+$ of tempered distributions with left-sided bounded support and the space $\mathcal{D}'_{F+}$ of distributions of finite order with left-sided bounded support. For the space $\mathcal{S}'_+$, problem (I) was studied in [**1, 4** and **6**].

We denote by $\mathbf{C}$ the complex plane and we set $\mathbf{C}_{\gtrless} = \{\zeta \in \mathbf{C}: |\operatorname{Im}\zeta| \gtrless 0\}$. We also write $D = i(d/d\xi)$. If n is a nonnegative integer, we denote by M_n the function defined on $\mathbf{R}$ by

$$M_n(\eta) = \begin{cases} 1 & \text{if } |\eta| > 1, \\ |\eta|^{-n} & \text{if } |\eta| \leq 1. \end{cases}$$

The spaces $\mathcal{D}'$ and $\mathcal{S}'$ are provided with the strong topologies.

1. Fourier transforms of distributions in $\mathcal{S}'_+$. Let U be a distribution in $\mathcal{S}'$ and suppose that there exists an analytic function u in C_- such that $u(\xi + i\eta) \to U_\xi$ in $\mathcal{S}'$ as $\eta \to 0-$. In that case we say that U can be continued in C_- to the analytic function u. The distribution U is called the boundary value of u on $\mathbf{R}$.

The following theorem is a slight modification of a theorem of H. G. Tillmann [**7**, Theorem 1.2]; we include the proof for the sake of completeness.

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THEOREM 1. *Let u be an analytic function in C_- which satisfies the estimate*

$$|u(\zeta)| \leq C(1 + |\zeta|)^m M_n(\eta) e^{a\eta}, \tag{1}$$

where $\zeta = \xi + i\eta \in \mathbf{C}_-$, m and n are positive integers, and a, C are constants. Then u has a distribution $U \in \mathcal{S}'$ as its boundary value on **R**.

PROOF. If u is an analytic function in $\mathbf{C}_-$ which satisfies (1), then so is the convolution

$$v(\zeta) = \int_{-\infty}^{\infty} u(\zeta - \tau)\varphi(\tau)\,d\tau,$$

where $\varphi \in \mathcal{D}$. Moreover, since

$$\frac{\partial^k v(\xi + i\eta)}{\partial \eta^k} = i^k \int_{-\infty}^{\infty} u(\xi + i\eta - \tau)\varphi^{(k)}(\tau)\,d\tau,$$

all derivatives $\partial^k v(\xi + i\eta)/\partial\eta^k$ satisfy (1) with the same integers m and n. But

$$\frac{\partial^k v(\xi + i\eta)}{\partial \eta^k} = \int_{-1}^{\eta} \frac{\partial^{k+1} v(\xi + i\tau)}{\partial \tau^{k+1}}\,d\tau + \left.\frac{\partial^k v(\xi + i\eta)}{\partial \eta^k}\right|_{\eta=-1}$$

and, therefore, one can reduce the integer n in the estimate (1) for v to 1 and next replace $|\eta|^{-1}$ by $\log|\eta|$. It follows that $v(\xi + i\eta)$ converges, as $\eta \to 0-$, uniformly on every compact subset of R, and the limit $V(\xi)$ is a C^∞-function which satisfies the growth conditions

$$|V^{(k)}(\xi)| \leq C_k(1 + |\xi|)^m, \qquad \xi \in \mathbf{R}, \tag{2}$$

where $k = 0, 1, \ldots,$ and C_k are constants. This shows, in particular, that $u(\xi + i\eta)$ converges in $\mathcal{D}'$ as $\eta \to 0-$. If U is the limit, then $V(\xi) = U * \varphi(\xi)$, and the estimates (2) imply that $U \in \mathcal{S}'$ (see [**5**, Theorem VI, p. 95]).

We now characterize Fourier transforms of distributions in $\mathcal{S}'_+$.

THEOREM 2. *T is a distribution in $\mathcal{S}'$ with support contained in the interval $[a, \infty)$ if and only if its Fourier transform $\hat{T}$ can be continued in C_- to an analytic function $\hat{t}$ satisfying the estimate*

$$|\hat{t}(\zeta)| \leq C(1 + |\zeta|)^m M_n(\eta) e^{a\eta}, \tag{3}$$

where $\zeta = \xi + i\eta$, m and n are postive integers, and C is a constant.

PROOF. If $T \in \mathcal{S}'$ and $\operatorname{supp} T \subset [a, \infty)$, then

$$T = D^m F, \tag{4}$$

where F is a continuous function on **R** such that $\operatorname{supp} F \subset [a, \infty)$ and $|F(x)| \leq C^*(1 + |x|)^{n-2}$; here m and n are positive integers and C^* is a constant. The function

$$\hat{f}(\zeta) = \int_{-\infty}^{\infty} F(x)e^{-ix\zeta}\,dx = \int_a^{\infty} F(x)e^{x\eta}e^{-ix\xi}\,dx$$

is obviously analytic in C_-, and we have

$$|\hat{f}(\zeta)| \leqslant \int_a^\infty |F(\zeta)| e^{x\eta}\, dx \leqslant C^* \sup_{a \leqslant x} (1 + |x|)^n e^{\eta x} \int_a^\infty \frac{dx}{(1+|x|)^2}$$

$$\leqslant \pi C^* \sup_{a \leqslant x} (1 + |x|)^n e^{\eta x}.$$

But

$$\sup_{a \leqslant x \leqslant |a|+n} (1+|x|)^n e^{\eta x} \leqslant (1 + |a| + n)^n e^{\eta a} \leqslant (2n + |a|)^n e^{\eta a},$$

and

$$\sup_{|a|+n \leqslant x} (1+|x|)^n e^{\eta x} \leqslant \begin{cases} (2n+|a|)^n e^{\eta a} & \text{if } |\eta| > 1, \\ (2n)^n e^{|a|} |\eta|^{-n} e^{\eta a} & \text{if } |\eta| \leqslant 1, \end{cases}$$

because $n \geqslant 1$. Hence,

(5) $$|\hat{f}(\zeta)| \leqslant \pi C^*(2n + |a|)^n e^{|a|} M_n(\eta) e^{a\eta}.$$

Since $Fe^{x\eta}$ converges in $\mathcal{S}'$ to F, as $\eta \to 0-$, the boundary value of $\hat{f}$ on $\mathbf{R}$ is $\hat{F}$.

If we now set $\hat{t}(\zeta) = (-\zeta)^m \hat{f}(\zeta)$, then $\hat{t}$ is an analytic function in C_- satisfying (3), because of (5). Its boundary value on $\mathbf{R}$ is $(-\xi)^m \hat{F} = \hat{T}$, by (4).

Conversely, suppose that $\hat{T}$ can be continued in C_- to an analytic function $\hat{t}$ satisfying (3). Then the distribution $U_\xi = e^{i(a+1)\xi}\hat{T}_\xi$ can be continued in C_- to the analytic function

(6) $$u(\zeta) = e^{i(a+1)\zeta}\hat{t}(\zeta),$$

which satisfies (1) with a replaced by -1.

Let ψ be an arbitrary function in $\mathcal{D}$ with $\operatorname{supp}\psi \subset (-\infty, 1]$. Its Fourier-Laplace transform $\hat{\psi}$ is an entire function with the following property: For any pair k, l of nonnegative integers, there exists a constant $C_{k,l}$ such that

(7) $$|D^k \hat{\psi}(\zeta)| \leqslant C_{k,l}(1 + |\zeta|)^{-l} e^{\eta} \quad \text{when } \eta \geqslant -2.$$

Now consider the convolution

(8) $$g(\zeta) = U * \hat{\psi}(\zeta) = \langle U_\tau, \hat{\psi}(\zeta - \tau) \rangle,$$

which is an entire function. We show that

(9) $$|g(\zeta)| \leqslant C_0(1 + |\zeta|)^m e^{|\eta|},$$

where C_0 is a constant.

Since $U \in \mathcal{S}'$, there are nonnegative integers p, q, a constant C_1, and a continuous function F on $\mathbf{R}$ such that

(10) $$U = D^p F \quad \text{and} \quad |F(\tau)| \leqslant C_1(1 + |\tau|)^q.$$

If $\eta \geqslant -2$, it follows from (7), (8) and (10) that

(11) $$|g(\zeta)| \leqslant \int_{-\infty}^\infty |F(\tau)| |D^p \hat{\psi}(\zeta - \tau)| \, d\tau \leqslant C_2(1 + |\zeta|)^q e^{\eta}$$

where $C_2 = \pi C_1 C_{p,q+2}$.

On the other hand, U can be continued in C_- to the analytic function u. Thus, for $\eta < -1$, we derive from (1), with $a = -1$, and (7) the estimate

$$|g(\zeta)| \leq \int_{-\infty}^{\infty} |u(\zeta - \tau)| |\hat{\psi}(\tau)| \, d\tau \leq C_3(1 + |\zeta|)^m e^{-\eta}, \tag{12}$$

where $C_3 = 2\pi C C_{0,m+2}$.

In (11) we can choose $q = m$ (with some constant C_2), since $|g(\zeta)|$ is bounded by $C_3 e^2(1 + |\zeta|)^m$ on the line $\eta = -2$. This, combined with (12), proves (9).

By the Paley-Wiener-Schwartz theorem, (9) implies that g is the Fourier-Laplace transform of a distribution S with support contained in the interval $[-1, 1]$. But, by (6) and (8),

$$S = 2\pi\psi_{\tau_{-a-1}}T,$$

where $_{\tau_{-a-1}}T$ is the translation of T by $-a - 1$, and ψ is an arbitrary function in $\mathcal{D}$ with $\operatorname{supp}\psi \subset (-\infty, 1]$. Hence, $\operatorname{supp}_{\tau_{-a-1}}T \subset [-1, \infty)$ and, therefore, $\operatorname{supp} T \subset [a, \infty)$. The theorem is now established.

We denote by $\mathcal{K}$ the collection of all analytic functions u in C_-, which satisfy (1), for some m, n, a and C. $\mathcal{K}$ is clearly an integral domain under addition and multiplication. We also denote by S'_+ the set of all distributions in $\mathcal{S}'$ which are boundary values of functions in $\mathcal{K}$. In view of Theorem 1, we can define in S'_+ a product in the following way.

DEFINITION 1. If u and v are functions in $\mathcal{K}$ whose boundary values are U and V, respectively, we set

$$U_\xi \circ V_\xi = \lim_{\eta \to 0-} u(\xi + i\eta) v(\xi + i\eta),$$

where the limit is taken in the topology of $\mathcal{S}'$.

If U and V are continuous functions of polynomial growth, then $U \circ V$ coincides with their usual product.

THEOREM 3. *If S and T are distributions in $\mathcal{S}'_+$, then*

$$(S * T)\hat{} = \hat{S} \circ \hat{T}. \tag{13}$$

PROOF. For $\eta < 0$, we have

$$((S_x e^{x\eta}) * (T_x e^{x\eta}))\hat{}(\xi) = \hat{s}(\xi + i\eta)\hat{t}(\xi + i\eta).$$

As $\eta \to 0-$, this implies (13).

We illustrate the product in S'_+ on one example.

EXAMPLE 1. If k is a nonnegative integer, consider the function

$$H_k(x) = \begin{cases} 0 & \text{for } x < 0, \\ x^k & \text{for } x \geq 0. \end{cases}$$

Its Fourier transform

$$\hat{H}_k = i^{k+1}\left[(-1)^{k+1} k!/\xi^{k+1} + i\pi\delta^{(k)}\right]$$

can be continued in $\mathbf{C}_-$ to the analytic function

$$\hat{h}(\zeta) = (-i)^{k+1} k!/\zeta^{k+1} = k!/(i\zeta)^{k+1}.$$

If l is another nonnegative integer, then

$$\hat{H}_k \circ \hat{H}_l = \left\{ i^{k+1}\left[\frac{(-1)^{k+1}k!}{\xi^{k+1}} + i\pi\delta^{(k)} \right] \right\} \circ \left\{ i^{l+1}\left[\frac{(-1)^{l}l!}{\xi^{l+1}} + i\pi\delta^{(l)} \right] \right\}$$

$$= \frac{i^{k+l+2}k!l!}{(k+l+1)!}\left[\frac{(-1)^{k+l+2}(k+l+1)!}{\xi^{k+l+2}} + i\pi\delta^{(k+l+1)} \right].$$

In particular, for the Heaviside function H_0,

$$\hat{H}_0 \circ \hat{H}_0 = [-i/\xi - \pi\delta] \circ [-i/\xi - \pi\delta] = -1/\xi^2 - i\pi\delta' = D\hat{H}_0.$$

As usual, δ denotes the Dirac measure with support at the origin.

2. Fourier transforms of distributions in $\mathfrak{D}'_{F+}$. Suppose that $T \in \mathfrak{D}'_F$. Then there exists an entire function $A(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $A(x) \neq 0$ for $x \in \mathbf{R}$ and

$$T = AT_0 = \sum_{n=0}^{\infty} a_n x^n T_0, \tag{14}$$

where $T_0 \in \mathfrak{S}'$ and the series converges in $\mathfrak{D}'_F$ (see e.g. **[8]**). It follows that the Fourier transform $\hat{T}$ of T can be represented by the formula

$$\hat{T} = A(D)\hat{T}_0 = \sum_{n=0}^{\infty} a_n D^n \hat{T}_0 \tag{15}$$

as a "derivative of infinite order" of the tempered distribution $\hat{T}_0$. The series in (15) converges in the topology of $\mathbf{D}'_F$.

Since $A(x) \neq 0$ on $\mathbf{R}$, $\operatorname{supp} T \subset [a, \infty)$ if and only if $\operatorname{supp} T_0 \subset [a, \infty)$. Thus, from Theorem 2, we obtain

THEOREM 4. *T is a distribution in $\mathfrak{D}'_F$ with support contained in $[a, \infty)$ if and only if its Fourier transform $\hat{T}$ can be represented in the form* (15), *where A is an entire function such that $A(x) \neq 0$ on $\mathbf{R}$ and $\hat{T}_0$ is a distribution in $\mathfrak{S}'$ which can be continued in $\mathbf{C}_-$ to an analytic function $\hat{t}_0$ satisfying* (3).

REMARK. In Theorem 4 one can require that $\hat{T}_0$ be a continuous function of polynomial growth, whose analytic continuation $\hat{t}_0$ in $\mathbf{C}_-$ satisfies the estimate $|\hat{t}_0(\zeta)| \leq C(1 + |\zeta|)^m e^{a\eta}$ for some integer m and some constant C.

If $U_0 \in S'_+$, it is clear that $D^k U_0 \in S'_+$, for all k. We now extend the definition of the product " $\circ$ " to the elements of $\mathbf{D}'_{F+}$.

DEFINITION 2. Suppose U and V are in $\mathbf{D}'_{F+}$. Then

$$U = A(D)U_0 = \sum_{k=0}^{\infty} a_k D^k U_0 \quad \text{and} \quad V = B(D)V_0 = \sum_{k=0}^{\infty} b_k D^k V_0, \tag{16}$$

where A, B are entire functions and U_0, V_0 are distributions in S'_+. We define the product of U and V by

$$U \circ V = \sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j b_{k-j} \left(D^j U_0 \right) \circ \left(D^{k-j} V_0 \right). \tag{17}$$

Each term of the series in (17) is a constant times a product of distributions in S'_+, as in Definition 1. In order to prove that Definition 2 is correct, we have to show that:

(a) the series in (17) converges in $\mathbf{D}'_F$;

(b) the sum of the series does not depend on the representation (16);

(c) the new product coincides with the one in Definition 1 if U and V are in S'_+.

We obtain properties (a)–(c) as a by-product of the proof of

THEOREM 5. *If S and T are distributions in $\mathcal{D}'_{F+}$, then*

$$(S * T)\hat{} = \hat{S} \circ \hat{T}. \tag{18}$$

PROOF. Suppose that

$$S = AS_0 = \sum_{k=0}^{\infty} a_k x^k S_0 \quad \text{and} \quad T = BT_0 = \sum_{k=0}^{\infty} b_k x^k T_0,$$

where A, B are entire functions and $S_0, T_0 \in \mathcal{S}'_+$. Then

$$\left(\sum_{k=0}^{n} a_k x^k S_0\right) * \left(\sum_{k=0}^{n} b_k x^k T_0\right) \to S * T$$

in $\mathcal{D}'_F$ as $n \to \infty$. But, by Theorem 3,

$$\left(\left(\sum_{k=0}^{n} a_k x^k S_0\right) * \left(\sum_{k=0}^{n} b_k x^k T_0\right)\right)\hat{} = \left(\sum_{k=0}^{n} a_k D^k \hat{S}_0\right) \circ \left(\sum_{k=0}^{n} b_k D^k \hat{T}_0\right)$$
$$= \sum_{k=0}^{n} \sum_{j=0}^{n} a_j b_{k-j} \left(D^j \hat{S}_0\right) \circ \left(D^{k-j} \hat{T}_0\right). \tag{19}$$

As $n \to \infty$, the limit of both sides in (19) exists in $\mathbf{D}'_F$ and we obtain (18).

EXAMPLE 2. Consider the function

$$E(x) = \begin{cases} 0 & \text{for } x < 0, \\ e^{\gamma x} & \text{for } x \geqslant 0, \end{cases}$$

where $\gamma > 0$. We can write

$$E(x) = e^{\gamma x} H_0(x) = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} x^k H(x),$$

where H_0 is the Heaviside function in Example 1. Hence,

$$\hat{E} = e^{\gamma D} \hat{H}_0 = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} D^k \hat{H}_0,$$

and, therefore,

$$\hat{E} \circ \hat{E} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left(\frac{\gamma^j}{j!} D^j \hat{H}_0\right) \circ \left(\frac{\gamma^{k-j}}{(k-j)!} D^{k-j} \hat{H}_0\right)$$
$$= \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} D^{k+1} \hat{H}_0 = D\hat{E}.$$

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Department of Mathematics, State University of New York, Buffalo, New York 14214