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DISAPPEARANCE OF EXTREME POINTS

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ABSTRACT. It is shown that every separable Banach space which contains an isomorphic copy of c_0 is isomorphic to a strictly convex space E such that no point of E is an extreme point of the unit ball of E^{**}.

An extreme point in the unit ball of a Banach space is *preserved* if it is extreme in the unit ball of the second dual (identifying the space with its natural embedding). In all the spaces $C(X)$, $L^p\,(1 \leq p \leq \infty)$, all extreme points are preserved. In 1961, R. R. Phelps [**2**] asked if there were any nonpreserved extreme points at all. The question was quickly answered by Katznelson (see note added in proof to [**2**]), who found a nonpreserved extreme point in the disk algebra.

Since a denting point of a set in E is a denting point of the weak* closure of the set in E^{**}, we see that: *if E has the Radon-Nikodym property then its unit ball has preserved extreme points* (see [**1**] for information about the Radon-Nikodym property). Of course, it is possible that *some* of the extreme points of a Radon-Nikodym space are not preserved.

We thank F. Sullivan and R. R. Phelps for helpful conversations on the subject of this paper.

Our result is

THEOREM. *Let E be a separable Banach space containing an isomorphic copy of c_0. Then E is isomorphic to a strictly convex space F such that no extreme point of F is preserved.*

We need two lemmas.

LEMMA 1. *Let E and G be Banach spaces and let $T\colon E \to G$ be a bounded linear operator. Define*

$$|||x||| = ||x|| + ||Tx||, \qquad x \in E.$$

Then

(i) $|||\cdot|||$ *is an equivalent norm on E;*

(ii) *the norm on $(E, |||\cdot|||)^{**}$ is given by*

$$|||x^{**}||| = ||x^{**}|| + ||T^{**}x^{**}||;$$

(iii) *if G is strictly convex and T is one-to-one then $|||\cdot|||$ is strictly convex.*

PROOF. (i) and (iii) are easy and well known.

(ii) Define $\Phi\colon (E, |||\cdot|||) \to H = (E \times G)_1$ by $\Phi(x) = (x, Tx)$, $x \in E$. Then Φ is a linear isometry onto the graph of T. Now $H^{**} = (E^{**} \times G^{**})_1$. Let us compute

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$\Phi^{**}: (E, |||\cdot|||)^{**} \to H^{**}$. Notice that the map $x^{**} \to (x^{**}, T^{**}x^{**})$ is continuous in the weak* topology. Since it and Φ^{**} agree on a weak* dense set (namely, E), we see that

$$\Phi^{**}(x^{**}) = (x^{**}, T^{**}x^{**}), \qquad x^{**} \in E^{**}.$$

Now Φ is an isometry and therefore so is Φ^{**}. Hence

$$|||x^{**}||| = |||\Phi^{**}x^{**}||| = ||x^{**}|| + ||T^{**}x^{**}||.$$

LEMMA 2. *There exists an infinite-dimensional weak*-closed subspace M of l^∞ such that $M \cap c_0 = \{0\}$.*

PROOF. The space l^2 is a quotient of l^1 (as is every separable space). Thus $l^2 \approx l^1/K$ and so $l^2 \approx (l^2)^* \approx K^\perp \subseteq l^\infty$. Now $K^\perp$ is weak*-closed and infinite dimensional. Its intersection with c_0 is at most one dimensional for the following reason: If, on the contrary, $c_0 \cap K^\perp$ had dimension > 1, then c_0 would contain an isometric copy of the Euclidean plane. It is well known that this is not possible. For one proof, observe that the dual of the Euclidean plane has uncountably many extreme points. Each of these would have an extension to an extreme point in $c_0^* = l^1$. But the latter space has only countably many extreme points.

Now take an appropriate hyperplane in $K^\perp$ to get M.

The proof of the theorem is divided into two steps for greater readability.

PROOF OF THE THEOREM. *Step* 1. Let $E = c_0$. Let us find a one-to-one operator T from c_0 to l^2 such that $T^{**}: l^\infty \to l^2$ has an infinite-dimensional null-space. To do this let $M \subseteq l^\infty$ be as in Lemma 2. Then $M = N^\perp$, where N is some closed subspace of l^1. Let $\{\varphi_i\}_1^\infty \subseteq N$ so that $N = \overline{\mathrm{sp}}\{\varphi_i\}_1^\infty$ and $\sum_1^\infty ||\varphi_i||^2 < +\infty$. Define

$$Tx = (\varphi_i(x)) \in l^2, \quad \forall x \in c_0.$$

Then

$$T^{**}x^{**} = (x^{**}(\varphi_1), \dots) \in l^2, \quad \forall x^{**} \in l^\infty.$$

We see that T is one-to-one and that the null-space of T^{**} is M. Now define

$$|||x||| = ||x|| + ||Tx||, \qquad x \in c_0.$$

Then $|||\cdot|||$ is strictly convex. Let $x \in c_0$ with $|||x||| = 1$. We show that x is not an extreme point in $(c_0, |||\cdot|||)^{**}$. Let m be an integer such that $|x_n| < \frac{1}{2}||x||$, $n \geq m$. (Here $x = (x_n)_1^\infty$.) Since M is infinite dimensional, there is a nonzero $y^{**} \in M$ with $(y^{**})_n = 0$ for $n < m$. Scale y^{**} so that $||y^{**}|| \leq \frac{1}{2}||x||$. Then $||x \pm y^{**}|| = ||x||$ and $||T^{**}(x \pm y^{**})|| = ||Tx||$. Hence

$$|||x \pm y^{**}||| = |||x||| = 1,$$

and so x is not extreme in $(c_0, |||\cdot|||)^{**}$.

Step 2. Now suppose E contains an isomorphic copy of c_0. Since E is separable, this copy is complemented (see [**1**, p. 178]). Thus E is isomorphic to $(G \times c_0)_\infty$, where the norm on the product space is

$$||(g, x)|| = \max(||g||, ||x||), \qquad g \in G,\ x \in c_0.$$

Let S be a bounded one-to-one operator from G to l^2. Then let $T: c_0 \to l^2$ be as in Step 1. Now the operator $U = S \times T: G \times c_0 \to (l^2 \times l^2)_2$ defined by $U(g, x) = (Sg, Tx)$ is one-to-one so that the norm $|||(g, x)||| = ||(g, x)|| + ||U(g, x)||$ is strictly convex. Let $(g, x) \in G \times c_0$ with $|||(g, x)||| = 1$. Choose, as in Step 1, $x^{**} \neq 0$ in l^∞ with $T^{**}x^{**} = 0$ and

$$||x \pm x^{**}|| = ||x|| \leq ||(g, x)||.$$

We have

$$\begin{aligned} |||(g, x) \pm (0, x^{**})||| &= ||(g, x \pm x^{**})|| + ||U(g, x \pm x^{**})|| \\ &= \max(||g||, ||x \pm x^{**}||) + ||U(g, x)|| \\ &\leq \max(||g||, ||x||) + ||U(g, x)|| = |||(g, x)||| = 1. \end{aligned}$$

It follows that (g, x) is not extreme.

There are two obvious questions raised by this result.

Question 1. Which spaces have the property that, under any renorming, some extreme point is preserved?

Question 2. Which spaces can be renormed to be strictly convex but to have no preserved extreme points?

The obvious conjecture for Question 1 is: *Radon-Nikodym.* The answer to Question 2 falls somewhere between: *Non-Radon-Nikodym* and *containing* c_0. It would be interesting to consider L^1 in this connection.

References

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