ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF RETARDED DIFFERENTIAL EQUATIONS

G. LADAS, Y. G. SFICAS¹ AND I. P. STAVROULAKIS²

Abstract. In this paper we obtain sufficient conditions under which every solution of the retarded differential equation

\[ x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0, \]

where \( \tau \) is a nonnegative constant, and \( p(t) > 0 \), is a continuous function, tends to zero as \( t \to \infty \). Also, under milder conditions, we prove that every oscillatory solution of (1) tends to zero as \( t \to \infty \). More precisely the following theorems have been established.

Theorem 1. Assume that \( \int_{t_0}^{\infty} p(t) \, dt = +\infty \) and \( \lim_{t \to \infty} \int_{t-r}^{t} p(s) \, ds < \pi/2 \) or \( \lim \sup_{t \to \infty} \int_{t-r}^{t} p(s) \, ds < 1 \). Then every solution of (1) tends to zero as \( t \to \infty \).

Theorem 2. Assume that \( \lim \sup_{t \to \infty} \int_{t-r}^{t} p(s) \, ds < 1 \). Then every oscillatory solution of (1) tends to zero as \( t \to \infty \).

1. Introduction and preliminaries. In the present paper we obtain sufficient conditions under which every solution of the retarded differential equation

\[ x'(t) + p(t)x(t - \tau) = 0, \quad t \geq t_0, \]

where \( \tau \) is a nonnegative constant and \( p(t) \) is a continuous function tends to zero as \( t \to \infty \). Also, under milder conditions, we prove that every oscillatory solution of (1) tends to zero as \( t \to \infty \).

It is known, see for example [1, p. 330], that every solution of the retarded differential equation

\[ y'(t) + py(t - \tau) = 0, \]

where

\[ p > 0 \quad \text{and} \quad 0 \leq p\tau < \pi/2, \]

tends to zero as \( t \to \infty \). This is because (3) implies that every root of the characteristic equation of (2) has negative real part.

More precisely, if \( y(t; t_0, \phi) \) denotes the solution of (2) satisfying the initial condition \( y_{t_0} = \phi \) where \( \phi \in C[-\tau, 0] \), then (see [1, p. 325]) there exist positive constants \( M \) and \( \gamma \) such that

\[ |y(t; t_0, \phi)| \leq M \|\phi\| e^{-\gamma(t-t_0)}, \quad t \geq t_0, \]
where
\[ \| \phi \| = \sup_{-\tau s \leq 0} | \phi(s) | . \]

Also, if \( z(t; t_0, 0) \) denotes the solution of
\[ z'(t) + p z(t - \tau) = h(t), \quad t \geq t_0, \]
with zero initial function at \( t_0 \), that is, \( z_{t_0} = 0 \), then (see [1, p. 336])
\[ | z(t; t_0, 0) | \leq \frac{M}{\gamma} e^{(p - \gamma) \tau} \max_{t_0 \leq s < t} | h(s) | . \]

The following lemma is needed in the proof of Theorem 1.

**Lemma.** Consider the retarded differential equation
\[ x'(t) + x(t - \sigma(t)) = 0, \quad t \geq t_0, \]
where \( 0 \leq \sigma(t) \leq t \) is continuous and \( \lim_{t \to \infty} \sigma(t) = \tau < \infty \) exists. Assume that
\[ \tau < \pi/2. \]
Then every solution of (7) tends to zero as \( t \to \infty \).

**Proof.** Let \( x(t) \) be any solution of (7). Choose \( t_1 \geq t_0 + 2\tau + 2 \) and also such that
\[ \sigma(t) \leq \tau + 1, \quad t \geq t_1, \]
and
\[ (M/\gamma) e^{(1 + \gamma) \tau} | \tau - \sigma(t) | \leq \frac{1}{2}, \quad t \geq t_1, \]
where the constants \( M \) and \( \gamma \) are as defined in (4) with \( p = 1 \). Let \( y(t) \) be the solution of
\[ y'(t) + y(t - \tau) = 0, \quad t \geq t_1, \]
with initial function \( y_{t_1} = x_{t_1} \). Then, in view of (8), \( y(t) \) tends to zero as \( t \to \infty \). Set
\[ z(t) = x(t) - y(t) \]
and observe that \( z(t) \) satisfies the equation
\[ z'(t) + z(t - \tau) = x(t - \tau) - x(t - \sigma(t)), \quad t \geq t_1, \]
with zero initial function at \( t_1 \). Using (6), with \( p = 1 \) and
\[ h(s) = x(s - \tau) - x(s - \sigma(s)), \]
we find
\[ | z(t) | \leq \frac{M}{\gamma} e^{(1 + \gamma) \tau} \max_{t_1 \leq s < t} | x(s - \tau) - x(s - \sigma(s)) | . \]

Applying the mean value theorem and equation (7), we obtain
\[ | x(s - \tau) - x(s - \sigma(s)) | = | \sigma(s) - \tau | | x'(\xi) | = | \sigma(s) - \tau | | x(\xi - \sigma(\xi)) | , \]
where \( \xi \) is between \( s - \tau \) and \( s - \sigma(s) \). Then, setting
\[ B_1 = \max_{t_0 \leq s \leq t_1} | x(s) | , \]
we find
\[
\max_{t_1 \leq s \leq t} |x(s - \tau) - x(s - \sigma(s))| \leq \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \cdot \max_{t_0 \leq s \leq t} |x(s)| 
\]
\[
\leq \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right].
\]

Thus (12) implies
\[
|x(t)| - |y(t)| \leq \frac{M}{\gamma} e^{(1+\gamma)\tau} \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right].
\]
and, in view of (10),
\[
|x(t)| \leq |y(t)| + \frac{1}{2} \left[B_1 + \max_{t_1 \leq s \leq T} |x(s)| \right].
\]
Hence for every \( T \geq t_1 \) and for \( t_1 \leq t \leq T \) we have
\[
|x(t)| \leq |y(t)| + \frac{1}{2} \left[B_1 + \max_{t_1 \leq s \leq T} |x(s)| \right],
\]
and taking the maximum of both sides and rearranging terms we find
\[
\max_{t_1 \leq s \leq T} |x(t)| \leq 2 \max_{t_1 \leq s \leq T} |y(t)| + B_1.
\]
That is, \( x(t) \) is a bounded function and so there exists a \( B \geq B_1 \) such that
\[
|x(t)| \leq B \quad \text{for } t \geq t_0.
\]
Thus (13) yields
\[
|x(t)| \leq |y(t)| + 2B \frac{M}{\gamma} e^{(1+\gamma)\tau} \max_{t_1 \leq s \leq t} |\sigma(s) - \tau|,
\]
which implies that
\[
\lim_{t \to \infty} x(t) = 0.
\]
The proof is complete.

Note. Professor Driver informed us that this lemma follows from known results in the stability theory of delay differential equations, namely, Theorem G [1, p. 394], by taking \( F(t, \psi) = -\psi(-\sigma(t)) \) and \( h(t, \psi) = -\psi(-\sigma(t)) + \psi(-\tau) \). However we presented our proof because it is simple and direct.

2. Main results.

Theorem 1. Consider the retarded differential equation (1) where \( \tau \) is a nonnegative constant and \( p(t) > 0 \) is continuous. Assume that
\[
\int_{t_0}^{\infty} p(t) \, dt = +\infty,
\]
\[
\lim_{t \to -\infty} \int_{t-\tau}^{t} p(s) \, ds \text{ exists, and}
\]
\[
\lim_{t \to -\infty} \int_{t-\tau}^{t} p(s) \, ds < \frac{\pi}{2}.
\]
Then every solution of (1) tends to zero as \( t \to \infty \).
Proof. Set
\[ u = \sigma(t) \equiv \int_{t_0}^{t} p(s) \, ds, \quad t \geq t_0, \]
and observe that \( \sigma^{-1} \) exists, \( \lim_{t \to \infty} u(t) = \infty \), and
\[
\sigma(t - \tau) = \int_{t_0}^{t} p(s) \, ds = \int_{t_0}^{t} p(s) \, ds - \int_{t - \tau}^{t} p(s) \, ds = u - \int_{\sigma^{-1}(u) - \tau}^{\sigma^{-1}(u)} p(s) \, ds.
\]
That is,
\[
t - \tau = \sigma^{-1}\left( u - \int_{\sigma^{-1}(u) - \tau}^{\sigma^{-1}(u)} p(s) \, ds \right).
\]
Then the transformation
\[
z(u) = x(\sigma^{-1}(u)),
\]
reduces (1) to
\[
(16) \quad z'(u) + z\left( u - \int_{\sigma^{-1}(u) - \tau}^{\sigma^{-1}(u)} p(s) \, ds \right) = 0.
\]
In view of condition (15), the hypotheses of the Lemma are satisfied for equation (16) and therefore \( \lim_{u \to \infty} z(u) = \lim_{t \to \infty} x(t) = 0 \). The proof of the theorem is complete.

The next theorem shows that we may relax the left-hand side of condition (15) by replacing limit by limit superior if we strengthen the right-hand side by replacing \( \pi/2 \) by 1.

Theorem 2. Consider the retarded differential equation
\[
x'(t) + p(t)x(t - \tau) = 0
\]
where \( \tau \) is a positive constant and \( p(t) > 0 \) a continuous function. Assume that
\[
(17) \quad \limsup_{t \to \infty} \int_{t - \tau}^{t} p(s) \, ds < 1.
\]
Then every oscillatory solution of (1) tends to zero as \( t \to \infty \).

Proof. Let \( x(t) \) be an oscillatory solution of (1) which does not tend to zero as \( t \to \infty \). Then there exists a sequence \( t_n, \ n = 1, 2, \ldots \), of zeros of \( x(t) \) with the property that \( t_{n+1} - t_n \geq \tau \) and \( x(t) \not\equiv 0 \) on \( (t_n, t_{n+1}) \) for \( n = 1, 2, \ldots \). Set
\[
 s_n = \max_{t_n \leq t \leq t_{n+1}} |x(t)|, \quad n = 1, 2, \ldots .
\]
It suffices to prove that the sequence \( s_n \) tends to zero as \( t \to \infty \). Observe that
\[
 s_n = |x(\xi_n)|, \quad n = 1, 2, \ldots ,
\]
for some \( \xi_n \in (t_n, t_{n+1}) \) and that \( x'(\xi_n) = 0 \). Hence, from (1) \( x(\xi_n - \tau) = 0 \). Set \( \tau_n = \max\{t_n, \xi_n - \tau\}, \ n = 1, 2, \ldots \). Integrating (1) from \( \tau_n \) to \( \xi_n \) we obtain
\[
(18) \quad x(\xi_n) = -\int_{\tau_n}^{\xi_n} p(s)x(s - \tau) \, ds.
\]
Since $\tau_n \leq s \leq \xi_n$, it follows that $t_{n-1} \leq s - \tau \leq t_{n+1}$ and so, from (18),
\[ |x(\xi_n)| \leq \int_{\tau_n}^{\xi_n} p(s) |x(s - \tau)| \, ds \leq \left( \max_{t_{n-1} \leq t \leq t_{n+1}} |x(t)| \right) \int_{\tau_n}^{\xi_n + \tau} p(s) \, ds \]
or
\[ s_n \leq \left( \max\{s_n, s_{n-1}\} \right) \int_{\tau_n}^{\xi_n + \tau} p(s) \, ds. \]

In view of (17) it follows that for sufficiently large $n$, say $n \geq n_0$,
\[ s_n \leq s_{n-1} \int_{\tau_n}^{\xi_n + \tau} p(s) \, ds. \]

Now choose a number $\mu$ such that
\[ \limsup_{t \to \infty} \int_{t-\tau}^{t} p(s) \, ds < \mu < 1. \]

Then for $N \geq n_0$ sufficiently large,
\[ \int_{\tau_n}^{\xi_n + \tau} p(s) \, ds \leq \mu < 1, \quad n \geq N, \]

and (19) yields
\[ s_n \leq \mu s_{n-1}, \quad n \geq N. \]

This implies that
\[ \lim_{n \to \infty} s_n = 0, \]
and the proof is complete.

If in addition to the hypotheses of Theorem 2 we assume that $\int_{t_0}^{\infty} p(s) \, ds = +\infty$, then it is easy to prove that every nonoscillatory solution of (1) tends to zero as $t \to \infty$. In fact, let $x(t)$ be a nonoscillatory solution of (1) which does not tend to zero as $t \to \infty$. Without loss of generality, we assume that for some $t_1 > t_0$, $x(t) > 0$. Thus for $t_2$ sufficiently large
\[ x(t) - x(t_2) + \int_{t_2}^{t} p(s) x(s - \tau) \, ds = 0, \]
and (because $x(t)$ decreases)
\[ x(t) - x(t_2) + x(t - \tau) \int_{t_2}^{t} p(s) \, ds \leq 0, \]
which, as $t \to \infty$, leads to a contradiction.

The above observation together with Theorem 2 leads to the following result.

**Theorem 3.** Consider the retarded differential equation (1) where $\tau$ is a nonnegative constant and $p(t) > 0$ a continuous function. Assume that $\int_{t_0}^{\infty} p(t) \, dt = +\infty$ and
\[ \limsup_{t \to \infty} \int_{t-\tau}^{t} p(s) \, ds < 1. \]

Then every solution of (1) tends to zero as $t \to \infty$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
3. Examples. When $p(t)$ in equation (1) is a positive constant, the hypothesis (15) of Theorem 1 becomes $p \tau < \pi/2$, which is the "best" condition for every solution of equation (2) to tend to zero as $t \to \infty$. Indeed, when $p \tau = \pi/2$, as for example in
\[(20) \quad y'(t) + y(t - \pi/2) = 0,
\]
some solutions (e.g., $y(t) = \sin t$ or $\cos t$) do not tend to zero as $t \to \infty$.

Observe that the hypothesis (17) of Theorem 2 is not satisfied by equation (20) and therefore it is not surprising that the oscillatory solutions $y(t) = \sin t$ or $\cos t$ of (20) do not tend to zero as $t \to \infty$.

Remark 1. Note that Theorem 2 describes the asymptotic behaviour of the oscillatory solutions only. Now the question is the following: Are there cases where all solutions of (1) oscillate and therefore Theorem 2 describes the asymptotic behavior of all solutions as Theorem 1? The answer to this question is yes and such conditions are described in [2], namely
\[
p(t) > 0, \quad \liminf_{t \to \infty} \int_{t - \pi/2}^{t} p(s) \, ds > 0,
\]
and
\[
\liminf_{t \to \infty} \int_{t - \pi}^{t} p(s) \, ds > \frac{1}{e}.
\]

Example 1. Consider the retarded differential equation
\[
x'(t) + p(2 + \cos t)x(t - 2\pi) = 0, \quad t \geq 0,
\]
where $0 < p < \frac{1}{8}$.

Observe that for $p(t) = p(2 + \cos t)$,
\[
\int_{0}^{\infty} p(t) \, dt = +\infty \quad \text{and} \quad \int_{t - 2\pi}^{t} p(s) \, ds = 4p\pi < \frac{\pi}{2},
\]
that is the hypotheses of Theorem 1 are satisfied. Therefore every solution of this equation tends to zero as $t \to \infty$.

Example 2. Consider the retarded differential equation
\[
x'(t) + p(2 + \cos t)x(t - \pi) = 0, \quad t \geq 0,
\]
where $0 < p < 1/2(\pi + 1)$.

Observe that for $p(t) = p(2 + \cos t)$
\[
\int_{0}^{\infty} p(t) \, dt = +\infty \quad \text{and} \quad \int_{t - \pi}^{t} p(s) \, ds = \int_{t - \pi}^{t} p(2 + \cos s) \, ds = 2p(\pi + \sin t).
\]

Moreover,
\[
\limsup_{t \to \infty} \int_{t - \pi}^{t} p(s) \, ds = 2p(\pi + 1) < 1
\]
and, therefore, the hypotheses of Theorem 3 are satisfied. Thus every solution of this equation tends to zero as $t \to \infty$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
EXAMPLE 3. If $\tau < \pi/2$ then, according to the Lemma, every solution of each of the following retarded differential equations

$$x'(t) + x(t - \tau - 1/t) = 0 \quad \text{and} \quad x'(t) + x(t - \tau - e^{-t}) = 0$$

tends to zero as $t \to \infty$.

REFERENCES


Department of Mathematics, University of Rhode Island, Kingston, Rhode Island 02881