PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 88, Number 2, June 1983

ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF RETARDED DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper we obtain sufficient conditions under which every solution of the retarded differential equation

(1) $$x'(t) + p(t)x(t-\tau) = 0, \qquad t \geq t_0,$$

where τ is a nonnegative constant, and $p(t) > 0$, is a continuous function, tends to zero as $t \to \infty$. Also, under milder conditions, we prove that every oscillatory solution of (1) tends to zero as $t \to \infty$. More precisely the following theorems have been established.

THEOREM 1. *Assume that* $\int_{t_0}^{\infty} p(t)\,dt = +\infty$ *and* $\lim_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds < \pi/2$ *or* $\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds < 1$. *Then every solution of* (1) *tends to zero as* $t \to \infty$.

THEOREM 2. *Assume that* $\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds < 1$. *Then every oscillatory solution of* (1) *tends to zero as* $t \to \infty$.

1. Introduction and preliminaries. In the present paper we obtain sufficient conditions under which every solution of the retarded differential equation

(1) $$x'(t) + p(t)x(t-\tau) = 0, \qquad t \geq t_0,$$

where τ is a nonnegative constant and $p(t)$ is a continuous function tends to zero as $t \to \infty$. Also, under milder conditions, we prove that every oscillatory solution of (1) tends to zero as $t \to \infty$.

It is known, see for example [**1**, p. 330], that every solution of the retarded differential equation

(2) $$y'(t) + py(t-\tau) = 0,$$

where

(3) $$p > 0 \quad \text{and} \quad 0 \leq p\tau < \pi/2,$$

tends to zero as $t \to \infty$. This is because (3) implies that every root of the characteristic equation of (2) has negative real part.

More precisely, if $y(t; t_0, \phi)$ denotes the solution of (2) satisfying the initial condition $y_{t_0} = \phi$ where $\phi \in C[-\tau, 0]$, then (see [**1**, p. 325]) there exist positive constants M and γ such that

(4) $$|y(t; t_0, \phi)| \leq M\|\phi\| e^{-\gamma(t-t_0)}, \qquad t \geq t_0,$$

Received by the editors July 30, 1982.
1980 *Mathematics Subject Classification*. Primary 34K25; Secondary 34K15.
Key words and phrases. Retarded differential equation, solutions tend to zero, oscillatory solution.

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0002-9939/82/0000-1097/$02.25

where

$$\|\phi\| = \sup_{-\tau \leq s \leq 0} |\phi(s)| .$$

Also, if $z(t;\, t_0, 0)$ denotes the solution of

$$z'(t) + pz(t-\tau) = h(t), \qquad t \geq t_0, \tag{5}$$

with zero initial function at t_0, that is, $z_{t_0} = 0$, then (see [**1**, p. 336])

$$|z(t;\, t_0, 0)| \leq \frac{M}{\gamma} e^{(p+\gamma)\tau} \max_{t_0 \leq s \leq t} |h(s)| . \tag{6}$$

The following lemma is needed in the proof of Theorem 1.

LEMMA. *Consider the retarded differential equation*

$$x'(t) + x(t - \sigma(t)) = 0, \qquad t \geq t_0, \tag{7}$$

where $0 \leq \sigma(t) \leq t$ *is continuous and* $\lim_{t \to \infty} \sigma(t) = \tau < \infty$ *exists. Assume that*

$$\tau < \pi/2. \tag{8}$$

Then every solution of (7) *tends to zero as* $t \to \infty$.

PROOF. Let $x(t)$ be any solution of (7). Choose $t_1 \geq t_0 + 2\tau + 2$ and also such that

$$\sigma(t) \leq \tau + 1, \qquad t \geq t_1, \tag{9}$$

and

$$(M/\gamma)e^{(1+\gamma)\tau} |\tau - \sigma(t)| \leq \tfrac{1}{2}, \qquad t \geq t_1, \tag{10}$$

where the constants M and γ are as defined in (4) with $p = 1$. Let $y(t)$ be the solution of

$$y'(t) + y(t-\tau) = 0, \qquad t \geq t_1,$$

with initial function $y_{t_1} = x_{t_1}$. Then, in view of (8), $y(t)$ tends to zero as $t \to \infty$. Set $z(t) = x(t) - y(t)$ and observe that $z(t)$ satisfies the equation

$$z'(t) + z(t-\tau) = x(t-\tau) - x(t - \sigma(t)), \qquad t \geq t_1, \tag{11}$$

with zero initial function at t_1. Using (6), with $p = 1$ and

$$h(s) = x(s-\tau) - x(s - \sigma(s)),$$

we find

$$|z(t)| \leq \frac{M}{\gamma} e^{(1+\gamma)\tau} \max_{t_1 \leq s \leq t} |x(s-\tau) - x(s-\sigma(s))| . \tag{12}$$

Applying the mean value theorem and equation (7), we obtain

$$|x(s-\tau) - x(s-\sigma(s))| = |\sigma(s) - \tau|\,|x'(\xi)| = |\sigma(s) - \tau|\,|x(\xi - \sigma(\xi))| ,$$

where ξ is between $s - \tau$ and $s - \sigma(s)$. Then, setting

$$B_1 = \max_{t_0 \leq s \leq t_1} |x(s)| ,$$

we find

$$\max_{t_1 \leq s \leq t} |x(s-\tau) - x(s-\sigma(s))| \leq \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \cdot \max_{t_0 \leq s \leq t} |x(s)|$$

$$\leq \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right].$$

Thus (12) implies

$$|x(t)| - |y(t)| \leq \frac{M}{\gamma} e^{(1+\gamma)\tau} \max_{t_1 \leq s \leq t} |\sigma(s) - \tau| \left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right], \tag{13}$$

and, in view of (10),

$$|x(t)| \leq |y(t)| + \frac{1}{2}\left[B_1 + \max_{t_1 \leq s \leq t} |x(s)| \right].$$

Hence for every $T \geq t_1$ and for $t_1 \leq t \leq T$ we have

$$|x(t)| \leq |y(t)| + \frac{1}{2}\left[B_1 + \max_{t_1 \leq s \leq T} |x(s)| \right],$$

and taking the maximum of both sides and rearranging terms we find

$$\max_{t_1 \leq s \leq T} |x(t)| \leq 2 \max_{t_1 \leq s \leq T} |y(t)| + B_1.$$

That is, $x(t)$ is a bounded function and so there exists a $B \geq B_1$ such that

$$|x(t)| \leq B \quad \text{for } t \geq t_0.$$

Thus (13) yields

$$|x(t)| \leq |y(t)| + 2B\frac{M}{\gamma} e^{(1+\gamma)\tau} \max_{t_1 \leq s \leq t} |\sigma(s) - \tau|,$$

which implies that

$$\lim_{t \to \infty} x(t) = 0.$$

The proof is complete.

Note. Professor Driver informed us that this lemma follows from known results in the stability theory of delay differential equations, namely, Theorem G [**1**, p. 394], by taking $F(t, \psi) = -\psi(-\sigma(t))$ and $h(t, \psi) = -\psi(-\sigma(t)) + \psi(-\tau)$. However we presented our proof because it is simple and direct.

2. Main results.

THEOREM 1. *Consider the retarded differential equation* (1) *where* τ *is a nonnegative constant and* $p(t) > 0$ *is continuous. Assume that*

$$\int_{t_0}^{\infty} p(t)\, dt = +\infty, \tag{14}$$

$\lim_{t \to \infty} \int_{t-\tau}^{t} p(s)\, ds$ *exists, and*

$$\lim_{t \to \infty} \int_{t-\tau}^{t} p(s)\, ds < \frac{\pi}{2}. \tag{15}$$

Then every solution of (1) *tends to zero as* $t \to \infty$.

PROOF. Set

$$u = \sigma(t) \equiv \int_{t_0}^{t} p(s)\, ds, \qquad t \geq t_0,$$

and observe that σ^{-1} exists, $\lim_{t\to\infty} u(t) = \infty$, and

$$\sigma(t-\tau) = \int_{t_0}^{t-\tau} p(s)\, ds = \int_{t_0}^{t} p(s)\, ds - \int_{t-\tau}^{t} p(s)\, ds = u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s)\, ds.$$

That is,

$$t - \tau = \sigma^{-1}\left(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s)\, ds \right).$$

Then the transformation

$$z(u) = x(\sigma^{-1}(u)),$$

reduces (1) to

$$z'(u) + z\left(u - \int_{\sigma^{-1}(u)-\tau}^{\sigma^{-1}(u)} p(s)\, ds \right) = 0. \tag{16}$$

In view of condition (15), the hypotheses of the Lemma are satisfied for equation (16) and therefore $\lim_{u\to\infty} z(u) = \lim_{t\to\infty} x(t) = 0$. The proof of the theorem is complete.

The next theorem shows that we may relax the left-hand side of condition (15) by replacing limit by limit superior if we strengthen the right-hand side by replacing $\pi/2$ by 1.

THEOREM 2. *Consider the retarded differential equation*

$$x'(t) + p(t)x(t-\tau) = 0 \tag{1}$$

where τ is a positive constant and $p(t) > 0$ a continuous function. Assume that

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\, ds < 1. \tag{17}$$

Then every oscillatory solution of (1) *tends to zero as $t \to \infty$.*

PROOF. Let $x(t)$ be an oscillatory solution of (1) which does not tend to zero as $t \to \infty$. Then there exists a sequence t_n, $n = 1, 2, \ldots$, of zeros of $x(t)$ with the property that $t_{n+1} - t_n \geq \tau$ and $x(t) \not\equiv 0$ on (t_n, t_{n+1}) for $n = 1, 2, \ldots$. Set

$$s_n = \max_{t_n \leq t \leq t_{n+1}} |x(t)|, \qquad n = 1, 2, \ldots.$$

It suffices to prove that the sequence s_n tends to zero as $t \to \infty$. Observe that

$$s_n = |x(\xi_n)|, \qquad n = 1, 2, \ldots,$$

for some $\xi_n \in (t_n, t_{n+1})$ and that $x'(\xi_n) = 0$. Hence, from (1) $x(\xi_n - \tau) = 0$. Set $\tau_n = \max\{t_n, \xi_n - \tau\}$, $n = 1, 2, \ldots$. Integrating (1) from τ_n to ξ_n we obtain

$$x(\xi_n) = -\int_{\tau_n}^{\xi_n} p(s)x(s-\tau)\, ds. \tag{18}$$

Since $\tau_n \leqslant s \leqslant \xi_n$, it follows that $t_{n-1} \leqslant s - \tau \leqslant t_{n+1}$ and so, from (18),

$$|x(\xi_n)| \leqslant \int_{\tau_n}^{\xi_n} p(s)\,|x(s-\tau)|\,ds \leqslant \Big(\max_{t_{n-1}\leqslant t\leqslant t_{n+1}} |x(t)|\Big)\int_{\tau_n}^{\tau_n+\tau} p(s)\,ds$$

or

$$s_n \leqslant (\max\{s_n, s_{n-1}\})\int_{\tau_n}^{\tau_n+\tau} p(s)\,ds.$$

In view of (17) it follows that for sufficiently large n, say $n \geqslant n_0$,

$$s_n \leqslant s_{n-1}\int_{\tau_n}^{\tau_n+\tau} p(s)\,ds. \tag{19}$$

Now choose a number μ such that

$$\limsup_{t\to\infty}\int_{t-\tau}^{t} p(s)\,ds < \mu < 1.$$

Then for $N \geqslant n_0$ sufficiently large,

$$\int_{\tau_n}^{\tau_n+\tau} p(s)\,ds \leqslant \mu < 1, \qquad n \geqslant N,$$

and (19) yields

$$s_n \leqslant \mu s_{n-1}, \qquad n \geqslant N.$$

This implies that

$$\lim_{n\to\infty} s_n = 0,$$

and the proof is complete.

If in addition to the hypotheses of Theorem 2 we assume that $\int_{t_0}^{\infty} p(s)\,ds = +\infty$, then it is easy to prove that every nonoscillatory solution of (1) tends to zero as $t \to \infty$. In fact, let $x(t)$ be a nonoscillatory solution of (1) which does not tend to zero as $t \to \infty$. Without loss of generality, we assume that for some $t_1 > t_0$, $x(t) > 0$. Thus for t_2 sufficiently large

$$x(t) - x(t_2) + \int_{t_2}^{t} p(s)x(s-\tau)\,ds = 0,$$

and (because $x(t)$ decreases)

$$x(t) - x(t_2) + x(t-\tau)\int_{t_2}^{t} p(s)\,ds \leqslant 0,$$

which, as $t \to \infty$, leads to a contradiction.

The above observation together with Theorem 2 leads to the following result.

THEOREM 3. *Consider the retarded differential equation* (1) *where* τ *is a nonnegative constant and* $p(t) > 0$ *a continuous function. Assume that* $\int_{t_0}^{\infty} p(t)\,dt = +\infty$ *and*

$$\limsup_{t\to\infty}\int_{t-\tau}^{t} p(s)\,ds < 1.$$

Then every solution of (1) *tends to zero as* $t \to \infty$.

3. Examples. When $p(t)$ in equation (1) is a positive constant, the hypothesis (15) of Theorem 1 becomes $p\tau < \pi/2$, which is the "best" condition for every solution of equation (2) to tend to zero as $t \to \infty$. Indeed, when $p\tau = \pi/2$, as for example in

$$y'(t) + y(t - \pi/2) = 0, \tag{20}$$

some solutions (e.g., $y(t) = \sin t$ or $\cos t$) do not tend to zero as $t \to \infty$.

Observe that the hypothesis (17) of Theorem 2 is not satisfied by equation (20) and therefore it is not surprising that the oscillatory solutions $y(t) = \sin t$ or $\cos t$ of (20) do not tend to zero as $t \to \infty$.

REMARK 1. Note that Theorem 2 describes the asymptotic behaviour of the oscillatory solutions only. Now the question is the following: Are there cases where all solutions of (1) oscillate and therefore Theorem 2 describes the asymptotic behavior of all solutions as Theorem 1? The answer to this question is yes and such conditions are described in [**2**], namely

$$p(t) > 0, \qquad \liminf_{t\to\infty} \int_{t-\tau/2}^{t} p(s)\,ds > 0,$$

and

$$\liminf_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > \frac{1}{e}.$$

EXAMPLE 1. Consider the retarded differential equation

$$x'(t) + p(2 + \cos t)x(t - 2\pi) = 0, \qquad t \geqslant 0,$$

where $0 < p < \frac{1}{8}$.

Observe that for $p(t) = p(2 + \cos t)$,

$$\int_0^\infty p(t)\,dt = +\infty \quad \text{and} \quad \int_{t-2\pi}^{t} p(s)\,ds = 4p\pi < \frac{\pi}{2},$$

that is the hypotheses of Theorem 1 are satisfied. Therefore every solution of this equation tends to zero as $t \to \infty$.

EXAMPLE 2. Consider the retarded differential equation

$$x'(t) + p(2 + \cos t)x(t - \pi) = 0, \qquad t \geqslant 0,$$

where $0 < p < 1/2(\pi + 1)$.

Observe that for $p(t) = p(2 + \cos t)$

$$\int_0^\infty p(t)\,dt = +\infty \quad \text{and} \quad \int_{t-\pi}^{t} p(s)\,ds = \int_{t-\pi}^{t} p(2 + \cos s)\,ds = 2p(\pi + \sin t).$$

Moreover,

$$\limsup_{t\to\infty} \int_{t-\pi}^{t} p(s)\,ds = 2p(\pi + 1) < 1$$

and, therefore, the hypotheses of Theorem 3 are satisfied. Thus every solution of this equation tends to zero as $t \to \infty$.

EXAMPLE 3. If $\tau < \pi/2$ then, according to the Lemma, every solution of each of the following retarded differential equations

$$x'(t) + x(t - \tau - 1/t) = 0 \quad \text{and} \quad x'(t) + x(t - \tau - e^{-t}) = 0$$

tends to zero as $t \to \infty$.

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