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# AN ERGODIC THEOREM FOR SEMIGROUPS OF CONTRACTIONS

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ABSTRACT. An ergodic theorem for semigroups of nonlinear contractions having precompact trajectories in a Banach space is proved.

**1. The main result.** Throughout this note $X$ will be a real Banach space, $C \subset X$ a closed subset of $X$ and $S(t)$, $t \geqslant 0$, a semigroup of contractions on $C$, that is a family of mappings $S(t)$: $C \to C$, $t \geqslant 0$, satisfying:

(i) $\lim_{t \to t_0} S(t)x = S(t_0)x$ for $t_0 \geqslant 0$, $x \in C$.

(ii) $S(t+s)x = S(t)S(s)x$ for $t, s \geqslant 0$, $x \in C$.

(iii) $\|S(t)x - S(t)y\| \leqslant \|x - y\|$ for $t \geqslant 0$, $x, y \in C$.

For $x \in C$ we denote by $\alpha(x) = \{S(t)x : t \geqslant 0\}$ the trajectory starting at $x$ and by

$$\omega(x) = \left\{ y : y = \lim_{t_n \to \infty} S(t_n)x, \text{ for some sequence } t_n \to \infty \right\},$$

the possibly empty $\omega$-limit set of $X$. If $\omega(x) \neq \varnothing$ then it follows from its definition that $\omega(x)$ is invariant under $S(t)$, $t \geqslant 0$, i.e. $S(t)$: $\omega(x) \to \omega(x)$ for $t \geqslant 0$ and

$$\lim_{t \to \infty} \operatorname{dist}(S(t)x, \omega(x)) = 0, \tag{1}$$

where $\operatorname{dist}(z, B)$ is the distance between the point $z$ and the set $B$. Assuming, as we will do below, that for some $x \in C$ the trajectory $\alpha(x)$ is precompact, it follows easily that $\omega(x)$ is nonempty and compact. In this case, $\omega(x)$ can be given the structure of a compact commutative group and the following much stronger assertion, which is our main result, holds.

THEOREM 1 (THE ERGODIC THEOREM). *Let $X, Y$ be real Banach spaces, $C \subset X$ be closed and let $S(t)$, $t \geqslant 0$, be a semigroup of contractions on $C$. If for some $x \in C$ the trajectory $\gamma(x)$ is precompact, then $\omega(x)$ is a compact commutative group, and for every $f$: $C \to Y$ which is uniformly on bounded subsets of $C$ we have*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T f(S(t)x)\, dt = \int_{\omega(x)} f(\xi)\, d\xi, \tag{2}$$

*where $d\xi$ is the unique normalized Haar's measure on $\omega(x)$.*

**2. The proof of Theorem 1.** Let $C \subset X$ be a closed subset of the Banach space $X$ and let $S(t)$, $t \geqslant 0$, be a semigroup of contractions on $C$. A subset $\Omega$ of $C$ is called *minimal* under $S(t)$, $t \geqslant 0$, if it is the closure of the trajectory $\gamma(y) = \{S(t)y : t \geqslant 0\}$

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for every $y \in \Omega$; it is *strongly invariant* under $S(t)$, $t \geqslant 0$, if for every $t \geqslant 0$, $S(t)$ is a homeomorphism of $\Omega$ onto itself so that $S(t)$, $t \geqslant 0$, can be extended as a continuous group on $\Omega$. The set $\Omega$ *is equi-almost periodic* under $S(t)$ if it is strongly invariant and for every $\varepsilon > 0$ the set of real numbers with the property $\sup_{y \in \Omega} \| S(t)y - y \| \leqslant \varepsilon$ is relatively dense. The following proposition, whose proof can be found for example in [**4**, Theorem 1], is a standard result from the theory of dynamical systems.

PROPOSITION 2. *If for some $x \in C$, $\omega(x) \neq \varnothing$, then $\omega(x)$ is minimal and strongly invariant under $S(t)$. For each $t \in \mathbf{R}$, $S(t)$ is an isometry on $\omega(x)$. Moreover, if $\omega(x)$ is compact, then it is equi-almost periodic under $S(t)$.*

The proof of Thereom 1 will follow easily from the next two lemmas.

LEMMA 3. *Let $F$: $C \to Y$ be uniformly continuous on bounded subsets of $C$. If the trajectory $\gamma(x)$ is bounded and $\omega(x) \neq \varnothing$ then*

$$\lim_{T \to \infty} \left\| \frac{1}{T} \int_0^T f(S(t)x)\, dt - \frac{1}{T} \int_0^T f(S(t)y)\, dt \right\| = 0 \quad \text{for every } y \in \omega(x). \tag{3}$$

The proof of this simple lemma is omitted. It can be found, e.g., in [**6**, Proposition 4.1].

If $\omega(x)$ is a nonempty compact subset of $C$ there is a standard way to endow it with a commutative groups structure (see e.g. [**5**, Theorem 8.16, p. 394]), and hence there is a unique normalized Haar measure on it which will be denoted by $d\xi$. Moreover, if $\omega(x) \neq \varnothing$ is compact it follows from Proposition 2 that it is a minimal set consisting of almost periodic motions, and hence by [**5**, Theorem 9.34, p. 510] we have

LEMMA 4. *Let $\omega(x) \neq \varnothing$ be compact and $y \in \omega(x)$. For every continuous real valued function $h$: $\omega(x) \to \mathbf{R}$ we have*

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T h(S(t)y)\, dt = \int_{\omega(x)} h(\xi)\, d\xi. \tag{4}$$

We turn now to the proof of Theorem 1.

PROOF OF THEOREM 1. From the precompactness of the trajectory $\gamma(x)$ it follows that it suffices to prove the theorem only for functions $f \in C(\omega(x); Y)$ (the space of continuous $Y$-valued functions on $\omega(x)$). Since $\omega(x)$ is compact each function $f \in C(\omega(x); Y)$ can be uniformly approximated by functions $g_n(z)$ of the form $g_n(z) = \sum_{k=1}^n h_k(z) e_k$ where $h_k(z)$: $\omega(x) \to \mathbf{R}$ is continuous and $e_k \in Y$ for $1 \leqslant k \leqslant n$. From Lemma 4 it follows readily that the theorem is true for functions $g_n$ of this form, and therefore by the uniform continuity of $f \in C(\omega(x); Y)$ and the functions $g_n \in C(\omega(x); Y)$, it is also true for any $f \in C(\omega(x); Y)$ and the proof is complete.

**3. Concluding remarks.** It is well known that if $A$ is an $m$-accretive operator in a Banach space $X$ (for the definitions and properties of such operators see e.g. [**1** and **3**]) then it generates a semigroup of contractions $S(t)$, $t \geqslant 0$, on $\overline{D(A)}$ given by the exponential formula

$$S(t)x = \lim_{n \to \infty} \left( I + \frac{t}{n} A \right)^{-n} x \quad \text{for } x \in \overline{D(A)}. \tag{5}$$

For the proof of (5) see [**3**].

The main assumption of the ergodic theorem is the precompactness of the trajectory $\gamma(x)$ for some $x \in C$. This condition is clearly satisfied for all $x \in C$ if the semigroup $S(t)$, $t \geq 0$, is compact for $t \geq 0$, i.e. for every $t > 0$, $S(t)$ is a compact operator. A characterization of such compact semigroups, in terms of their $m$-accretive generator, is given in [**2**].

The compactness of the semigroup $S(t)$, $t \geq 0$, is of course not necessary for the precompactness of all the trajectories of $S(t)$, $t \geq 0$. It is shown in [**4**, Theorem 3] that if $A$ is $m$-accretive, 0 is in the range of $A$ and the everywhere defined contractions $(I + tA)^{-1}$ are compact for all $t > 0$, then all the trajectories $S(t)$, $t \geq 0$, are precompact and thus one can apply Theorem 1 to such semigroups.

Finally we note that Theorem 1 is an extension of a similar result in [**6**, Theorem 4.5] which deals with the special case where $X$ is a real Hilbert space. The conditions there assure that $\omega(x)$ lies in a finite-dimensional subspace of $X$ and it is nonempty and bounded. Hence $\omega(x)$ is clearly compact and the situation is similar to that of Theorem 1.

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