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CONDITIONS FOR SOME POLYGONAL FUNCTIONS TO BE BAZILEVIČ

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ABSTRACT. Univalent functions in the disc whose image is a particular eight-sided polygonal region determined by two parameters are studied. Whether such a function is Bazilevič is determined in terms of the two parameters, and the set of real α's is specified such that the function is (α, β) Bazilevič for some β. For any interval $[a, b]$ where $1 < a \leqslant 3 \leqslant b$, a function of this type which is $(\alpha, 0)$ Bazilevič precisely when α is in this interval is found. Examples are given of non-Bazilevič functions with polygonal images and Bazilevič functions which are $(\alpha, 0)$ Bazilevič for a single value α.

Introduction and notation. For $\alpha > 0$ and β real, let $B(\alpha, \beta)$ be the class of (α, β) Bazilevič functions introduced in [2]. Sheil-Small [7] established that $\{(\alpha, \beta) \mid \alpha > 0, f \in B(\alpha, \beta)\}$ is a closed, convex subset of the half-plane $\alpha > 0$. Campbell and Pearce [3] refer to this set as the representation projection of the function, and we will refer to its intersection with the α-axis as the α-projection which, if nonempty, is a point or an interval. In this paper we look at the α-projections for a two parameter family of functions $f(z; \psi_1, \psi_2)$ mapping $|z| < 1$ conformally onto the interior of a polygon whose geometry is determined by the two parameters. The α-projection is given in terms of the endpoints of the interval as functions of the parameters ψ_1, ψ_2. Those $f(z; \psi_1, \psi_2)$ for which $\psi_1 > 2\psi_2$ are not in any $B(\alpha, \beta)$; i.e., we also have a subfamily consisting of non-Bazilevič functions. The type of non-Bazilevič function of Plaster [6] is a limiting function of the above subfamily. Our family also includes bounded functions whose α-projection is a single point.

We now describe the functions $f(z; \psi_1, \psi_2)$. Let $f(z; \psi_1, \psi_2)$ be the unique univalent analytic function in $|z| < 1$ (normalized $f(0) = 0$; $f'(0) > 0$) with continuation to $|z| = 1$ having a closed polygonal image $A_0 A_1 A_2 \cdots A_8$, $A_0 = A_8$ (Figure 1) determined by the angle parameters

$$(1) \qquad 0 < \psi_1 < \pi/2, \qquad 0 < \psi_2 < \pi/4$$

as follows:

$$(2) \qquad A_5 = -1 - i; \quad A_6 = 1 - i; \quad A_7 = 1 + i; \quad A_8 = A_0 = -1 + i;$$
$$A_1 = -1 + i \tan \psi_2; \quad A_4 = \overline{A_1}; \quad \arg A_2 = 3\pi/4;$$
$$\arg(A_2 - A_1) = \psi_1; \quad A_3 = \overline{A_2}.$$

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(The figure suggests that we refer to $f(z; \psi_1, \psi_2)$ satisfying (1) and (2) as a keyhole function.) The exterior angles are denoted $e_k = \pi - \angle A_k$ in the usual manner (see Nehari [5, p. 188]). Hence, when $\angle A_k < \pi$, then $0 < e_k < \pi$, and when $\angle A_k > \pi$, $-\pi < e_k < 0$. The exterior angles have values

$$e_1 = e_4 = \pi/2 + \psi_1; \quad e_2 = e_3 = -\pi/2 - \psi_1; \tag{3}$$
$$e_5 = e_6 = e_7 = e_8 = e_0 = \pi/2.$$

A remark is appropriate about two of the cases for the limiting values of the parameters ψ_1, ψ_2. When $0 < \psi_2 < \pi/4$ and $\psi_1 \to 0$, the limit function is close-to-convex. For $0 < \psi_1 < \pi/2$ and $\psi_2 \to 0$, the limit function may be thought of as the type of non-Bazilevič function described by Plaster [6].

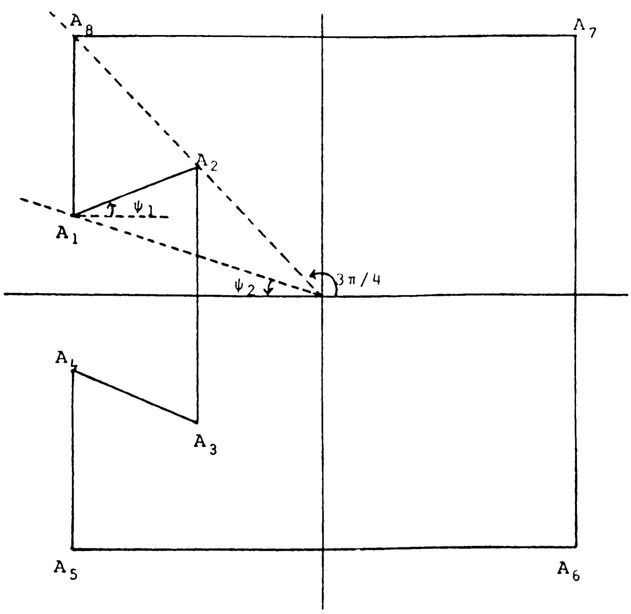

FIGURE 1

Some preliminary results. We will need the following theorem of Sheil-Small.

THEOREM 1 (SHEIL-SMALL). *If $f(z) \in B(\alpha, \beta)$ then*

(4)
$$\int_{\theta_1}^{\theta_2} \left\{ 1 + \operatorname{Re} \frac{re^{i\theta} f''(re^{i\theta})}{f'(re^{i\theta})} + (\alpha - 1) \operatorname{Re} \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} - \beta \operatorname{Im} \frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})} \right\} d\theta > -\pi,$$

for $0 < r < 1$ and $0 < \theta_2 - \theta_1 < 2\pi$.

Conversely, if $f(z)$ is analytic in $|z| < 1$, with $f(0) = 0$, $f(z) \neq 0$ $(0 < |z| < 1)$ and $f'(z) \neq 0$ $(|z| < 1)$, and if $f(z)$ satisfies (4) for $0 < r < 1$ and $0 < \theta_2 - \theta_1 < 2\pi$ when $\alpha \geq 0$ and β is real, then $f(z)$ is univalent in $|z| < 1$ and is in $B(\alpha, \beta)$ for $\alpha > 0$.

For convenience we denote the integrand of (4) as $F(f, \alpha, \beta; z)$, and the entire left side integral as $I_r(\theta_1, \theta_2)$. We define, additionally, a function $J(\theta_1, \theta_2)$ for $0 \leq \theta_2 - \theta_1 \leq 2\pi$:

$$J(\theta_1, \theta_2) = \liminf_{\substack{r \to 1^- \\ \mu_1 \to \theta_1 \\ \mu_2 \to \theta_2}} I_r(\mu_1, \mu_2). \tag{5}$$

Next we note a lemma which is needed to show that the representation projection of a keyhole function is symmetric in the α-axis.

LEMMA 1. *If $f \in B(\alpha, \beta)$ then $\bar{f} \in B(\alpha, -\beta)$.*

PROOF. Using the elementary properties of conjugates, we see that $F(\bar{f}, \alpha, -\beta; z) = F(f, \alpha, \beta; \bar{z})$ where $\bar{f}(z) = \overline{f(\bar{z})}$. The lemma now follows from Theorem 1.

The next lemma shows that if the Sheil-Small condition with weak inequality holds in the limit as $r \to 1^-$, the functions $f(z; \psi_1, \psi_2)$ are Bazilevič.

LEMMA 2. *If $f(z)$ is analytic in $|z| < 1$, with $f(0) = 0$, $f(z) \neq 0$ $(0 < |z| < 1)$, $f'(z) \neq 0$ $(|z| < 1)$, and $J(\theta_1, \theta_2) \geq -\pi$ for $0 \leq \theta_2 - \theta_1 \leq 2\pi$, then $f(z)$ is Bazilevič.*

PROOF. For fixed $\phi \in \mathbf{R}$ define

$$G_\phi(re^{i\tau}) = I_r(\phi - \tau, \phi + \tau) = \int_{-\tau}^{\tau} F(f, \alpha, \beta; re^{i(t+\phi)})\, dt,$$

where $0 < r < 1$, $0 < \tau < \pi$. The function G_ϕ is harmonic. One way to see this is to compute the Laplacian in polar coordinates, differentiate under the integral sign, and then use the fact that the integrand is harmonic as a function of r and t (see Baernstein [**1**, p. 153]). Clearly $I_r(\theta_1, \theta_2) > -\pi$ for all $0 < \theta_2 - \theta_1 < 2\pi$ if and only if $G_\phi(re^{i\tau}) > -\pi$ for all $\phi \in R$, $0 < r < 1$, and $0 < \tau < \pi$. The latter condition holds by the maximum principle for harmonic functions (Conway [**4**, p. 254]) when we show that for any fixed ϕ, and all a in the boundary of the upper half disc, $\liminf_{z \to a} G_\phi(z) \geq -\pi$.

We examine this limit in each of the four cases: $|a| = 1$; $0 < a < 1$; $-1 < a < 0$; $a = 0$. For $a = e^{i\tau}$ we have

$$\liminf_{z \to a} G_\phi(z) = \liminf_{\substack{r \to 1^- \\ \mu \to \tau}} G_\phi(re^{i\mu}) \geq J(\phi - \tau, \phi + \tau) \geq -\pi,$$

where the first inequality is by the definition of J and the second by hypothesis. If $0 < a < 1$,

$$\liminf_{z \to a} G_\phi(z) = \lim_{\substack{r \to a \\ \tau \to 0}} G_\phi(re^{i\tau}) = \int_\phi^\phi F(f, \alpha, \beta; re^{it})\, dt = 0 > -\pi.$$

If $-1 < a < 0$, using the Mean Value Theorem for harmonic functions (Conway [**4**, p. 253]) and the fact that $F(f, \alpha, \beta; 0) = \alpha$, we have

$$\liminf_{z \to a} G_\phi(z) = \lim_{\substack{r \to -a \\ \tau \to \pi}} G_\phi(re^{i\tau}) = \int_{\phi - \pi}^{\phi + \pi} F(f, \alpha, \beta; re^{it})\, dt = 2\pi\alpha > 0 > -\pi.$$

If $a = 0$,

$$\liminf_{z \to a} G_\phi(re^{i\tau}) = \inf_{0 \leqslant \tau \leqslant \pi} 2\tau\alpha = 0 > -\pi.$$

This completes the proof of the lemma.

The main theorem. We now determine the range of α's for which $f(z; \psi_1, \psi_2)$ is $(\alpha, 0)$ Bazilevič.

THEOREM 2. *$f(z; \psi_1, \psi_2) \in B(\alpha, 0)$ if and only if*

$$\psi_1/\psi_2 \leqslant \alpha - 1 \leqslant (\pi/2 - \psi_1)/(\pi/4 - \psi_2). \tag{6}$$

Also, $f(z; \psi_1, \psi_2)$ is Bazilevič (for some (α, β), $\alpha > 0$) if and only if $\psi_1 \leqslant 2\psi_2$.

PROOF. For keyhole $f(z; \psi_1, \psi_2)$, let $J(\theta_1, \theta_2)$ be defined as in (5). We note that if neither $f(e^{i\theta_1})$ nor $f(e^{i\theta_2})$ is a vertex of the polygon, then the limit actually exists in (5) and

$$J(\theta_1, \theta_2) = (\alpha - 1)(\arg f(e^{i\theta_2}) - \arg f(e^{i\theta_1})) + \sum e_k, \tag{7}$$

where $\arg f(re^{i\theta})$ is defined continuously for $\theta_1 \leqslant \theta \leqslant \theta_2$, and the summation is over all k for which the vertex A_k is on the curve $f(e^{i\theta})$, $\theta_1 < \theta < \theta_2$. By an analysis near points whose image is a vertex, we can verify that (7) also holds if $f(e^{i\theta_1})$ or $f(e^{i\theta_2})$ is a vertex, provided we take the sum over all k with A_k on $f(e^{i\theta})$, $\theta_1 \leqslant \theta \leqslant \theta_2$, replacing e_k by $\min\{0, e_k\}$ if $A_k = f(e^{i\theta_1})$ or $A_k = f(e^{i\theta_2})$.

We see that $\partial J/\partial\theta_2$ is positive when $f(e^{i\theta_2})$ is on the open segment $\overline{A_{j-1}A_j}$ and $j = 1, 3, 5, 6, 7, 8$ and negative when $f(e^{i\theta_2})$ is on the segment for $j = 2, 4$. Also for fixed θ_1, if $f(e^{i\omega}) = A_k$, the jump of $J(\theta_1, \theta_2)$ at $\theta_2 = \omega$ is e_k. Since J is lower semicontinuous, when we examine the signs of the e_k's as in (3), we see that for fixed θ_1, $J(\theta_1, \theta_2)$ may have a local minimum only when $f(e^{i\theta_2}) = A_2$ or A_4. A similar analysis gives that for θ_2 fixed, $J(\theta_1, \theta_2)$ may have a local minimum only when $f(e^{i\theta_1}) = A_1$ or A_3. Now suppose ω_k is such that $f(e^{i\omega_k}) = A_k$ for $0 \leqslant \omega_k \leqslant 2\pi$ for $k = 1, \dots, 8$. Then we have $\min_{0 \leqslant \theta_2 - \theta_1 \leqslant 2\pi} J(\theta_1, \theta_2)$ is the minimum of the four numbers $J(\omega_1, \omega_2)$, $J(\omega_1, \omega_4)$, $J(\omega_3, \omega_4)$ and $J(\omega_3, \omega_2 + 2\pi)$. We now compute, using (7),

$$\begin{aligned} J(\omega_1, \omega_2) &= e_2 + (\alpha - 1)(3\pi/4 - (\pi - \psi_2)) \\ &= -\pi/2 - \psi_1 + (\alpha - 1)(-\pi/4 + \psi_2), \\ J(\omega_1, \omega_4) &= e_2 + e_3 + (\alpha - 1)((\pi + \psi_2) - (\pi - \psi_2)) \\ &= -\pi - 2\psi_1 + (\alpha - 1)(2\psi_2), \end{aligned}$$

$$J(\omega_3, \omega_2 + \pi) = \sum_{j=1}^{8} e_j + (\alpha - 1)\left(\frac{11\pi}{4} - \frac{5\pi}{4}\right) = 2\pi + (\alpha - 1)\left(\frac{3\pi}{2}\right),$$

$$J(\omega_3, \omega_4) = J(\omega_1, \omega_2).$$

Now since $J(\omega_3, \omega_2 + \pi) > -\pi$, the minimum of the four numbers above is greater than or equal to $-\pi$ if and only if $J(\omega_1, \omega_2) \geqslant -\pi$ and $J(\omega_1, \omega_4) \geqslant -\pi$. These two inequalities yield relationship (6) of the theorem, and thus the first part of the theorem follows from Theorem 1 and Lemma 2.

To see the second part of the theorem, note that if $f(z;\psi_1,\psi_2)\in B(\alpha,\beta)$ then, by Lemma 1 and symmetry, $f(z;\psi_1,\psi_2)\in B(\alpha,-\beta)$. Since $\{(\alpha,\beta)\mid f(z)\in B(\alpha,\beta)\}$ is convex, this implies $f(z)\in B(\alpha,0)$. Thus, by the first part of the theorem we must have $((\pi/2-\psi_1)/(\pi/4-\psi_2))-\psi_1/\psi_2$ nonnegative, which holds if and only if $2\psi_2\geqslant\psi_1$.

Some remarks about the α-projection. The second part of our main theorem establishes that if $\psi_1\leqslant 2\psi_2$ then $f(z;\psi_1,\psi_2)$ has nonempty α-projection $[a,b]$, $a\leqslant b$, where

$$a=(\psi_1/\psi_2)+1,\qquad b=((\pi/2-\psi_1)/(\pi/4-\psi_2))+1. \tag{8}$$

For $\psi_1\leqslant 2\psi_2$, using (8), we see $a\leqslant 3\leqslant b$. Thus we have

COROLLARY 1. *$f(z;\psi_1,\psi_2)\in B(\alpha,\beta)$ for some (α,β) only if $f(z;\psi_1,\psi_2)\in B(3,0)$.*

COROLLARY 2. *The α-projection of a keyhole function is $\alpha=3$ if and only if $\psi_1=2\psi_2$.*

From (8) we also note that if $a=3$ then $b=3$, so the only intervals which may be α-projections for keyhole functions satisfy $a\leqslant 3\leqslant b$. Conversely, given an interval $[a,b]$, $a\leqslant 3\leqslant b$, we can exhibit the condition on ψ_1,ψ_2, hence, the specific keyhole function, for which a given interval is the α-projection.

COROLLARY 3. *The intervals $[a,b]$, $a\neq b$, which are α-projections of keyhole Bazilevič functions satisfy $1<a<3$, $b>3$. Every such interval is the α-projection of the $f(z;\psi_1,\psi_2)$ parameterized by*

$$\psi_1=\frac{\pi}{4}((a-1)(b-3))/(b-a),\quad \psi_2=\psi_1/(a-1). \tag{9}$$

The equations (9) follow directly from (8).

Specification of allowable values for any two of ψ_1,ψ_2,a,b as used above allows for computation of the other two using (8) and (9). For example, the interval form when $0<\psi_1=\psi_2<\pi/4$ is $[2,3+(\psi_1/(\pi/4-\psi_1))]$, so for $\psi_1=\psi_2=\pi/8$, the interval for which the function is $(\alpha,0)$ Bazilevič is $[2,4]$ and, in general, as $\psi_1=\psi_2\to\pi/4$, then $b\to\infty$, and as $\psi_1=\psi_2\to 0$, then $b\to 3$.

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