TENSOR PRODUCTS OF PRECLOSED OPERATORS
ON C*-ALGEBRAS

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Abstract. In this paper, we prove the following result: If $A_1$, $A_2$ are C*-algebras, and $T_1$, $T_2$ are preclosed operators on $A_1$, $A_2$ respectively, then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes \min A_2$. Furthermore, we show that the injective C*-cross norm $\| \cdot \|_{\min}$ is reflexive on the algebraic tensor product $A_1 \otimes A_2$.

Since Turumaru [8] introduced tensor products of C*-algebras, mysterious properties of C*-cross norms received much attention from many specialists. For example, Takesaki [6] found out that the C*-norm on the algebraic tensor product is not unique. Furthermore, Okayasu [3] showed that the minimal C*-norm is not a uniform cross norm. Therefore, it is natural to ask the question: when $\sigma$ and $\tau$ are bounded operators on C*-algebras $A_1$ and $A_2$, is $\sigma \otimes \tau$ bounded on $A_1 \otimes \min A_2$? The answer is negative. In 1970, Okayasu [3] gave an example of $\sigma$, $\tau$ and C*-algebras $A_1$, $A_2$ in such a way that $\sigma \otimes \tau$ is unbounded on $A_1 \otimes \min A_2$. Naturally, we will ask the question: what can we say about $\sigma \otimes \tau$ when $\sigma$ and $\tau$ are bounded? What kind of properties of the operators $\sigma$ and $\tau$ are preserved under the tensor product operation?

The purpose of this paper is to answer these questions. (1) If $\sigma$ and $\tau$ are bounded, then $\sigma \otimes \tau$ is preclosed on $A_1 \otimes \min A_2$. (2) If $\sigma$ and $\tau$ are densely defined and preclosed, then $\sigma \otimes \tau$ is densely defined and preclosed on $A_1 \otimes \min A_2$.

In Theorem 3, we show that the minimal C*-cross norm is reflexive.

Lemma 1. Let $E$ be a Banach space and $T$ be a densely defined linear operator on $E$. Then the following statements are equivalent.

1. $T$ is preclosed.
2. $T$ has a minimal closed linear extension; i.e., there exists a closed linear extension $\bar{T}$ of $T$ such that any closed linear extension of $T$ is a closed linear extension of $\bar{T}$.
3. For any $y \neq 0$ in $E$, $(0, y)$ is not in the closure of the graph of $T$.
4. $\mathcal{D}(T^*)$ is total in $E^*$.
5. $\mathcal{D}(T^*)$ is $\sigma(E^*, E)$-dense in $E^*$.

($T^*$ is the conjugate operator of $T$; $E^*$ is the conjugate space of $E$.)

Since we can find the proof of Lemma 1 in general Banach space text books, we omit it.
Lemma 2. Let $A_1, A_2$ be $C^*$-algebras and $A_1^*$ and $A_2^*$ be the conjugate spaces of $A_1, A_2$, respectively. Then the algebraic tensor product

$$F = A_1^* \otimes A_2^*$$

is $\sigma((A_1 \otimes_{\min} A_2)^*, A_1 \otimes_{\min} A_2)$-dense in $(A_1 \otimes_{\min} A_2)^*$.

Proof. From p. 208 in [7], if $f_1 \in A_1^*$, $f_2 \in A_2^*$ and $x = \sum_{i=1}^n x_{1,i} \otimes x_{2,i} \in A_1 \otimes A_2$, then

$$|\langle x, f_1 \otimes f_2 \rangle| \leq \|x\|_{\min} \|f_1\| \|f_2\|.$$

Thus, we have $A_1^* \otimes A_2^* \subseteq (A_1 \otimes_{\min} A_2)^*$.

To be $\sigma((A_1 \otimes_{\min} A_2)^*, A_1 \otimes_{\min} A_2)$-dense in $(A_1 \otimes_{\min} A_2)^*$ is equivalent to being total in $(A_1 \otimes_{\min} A_2)^*$ so we have to prove that $F$ is total in $(A_1 \otimes_{\min} A_2)^*$.

Suppose $x \in A_1 \otimes_{\min} A_2$ and $\langle x, f \rangle = 0$, $\forall f \in F$. We shall show $x = 0$.

Let $\sigma(A_1)$ and $\sigma(A_2)$ be the state spaces of $A_1$ and $A_2$, respectively, $\omega_1 \in \sigma(A_1)$ and $\omega_2 \in \sigma(A_2)$.

Let $\Pi_{\omega_1}$ and $\Pi_{\omega_2}$ be the cyclic representations corresponding to $\omega_1$ and $\omega_2$ with representation Hilbert spaces $\mathcal{G}_{\omega_1}$ and $\mathcal{G}_{\omega_2}$ respectively. We construct the tensor representation of $\Pi_{\omega_1}$ and $\Pi_{\omega_2}$,

$$\Pi_{\omega} = \Pi_{\omega_1} \otimes \Pi_{\omega_2}.$$

Therefore, for all $\xi_1, \eta_1 \in \mathcal{G}_{\omega_1}$ and $\xi_2, \eta_2 \in \mathcal{G}_{\omega_2}$,

$$(\Pi_{\omega}(x)(\xi_1 \otimes \xi_2) | \eta_1 \otimes \eta_2) = \langle x, f \otimes g \rangle = 0,$$

in which $\langle x_1, f \rangle = (\Pi_{\omega_1}(x_1) \xi_1 | \eta_1)$ and $\langle x_2, g \rangle = (\Pi_{\omega_2}(x_2) \xi_2 | \eta_2)$. Then

$$\left(\Pi_{\omega}(x)(\xi_1 \otimes \xi_2) | \sum_{j=1}^m \eta_{1,j} \otimes \eta_{2,j}\right) = 0.$$

Hence $\Pi_{\omega}(x)(\xi_1 \otimes \xi_2) \perp \mathcal{G}_{\omega_1} \otimes \mathcal{G}_{\omega_2}$.

Since $\mathcal{G}_{\omega_1} \otimes \mathcal{G}_{\omega_2}$ is dense in $\mathcal{G}_{\omega}$, $\Pi_{\omega}(x)(\xi_1 \otimes \xi_2) = 0$; also

$$\Pi_{\omega}(x) \left(\sum_{i=1}^n \xi_{1,i} \otimes \xi_{2,i}\right) = 0, \quad \Pi_{\omega}(x) = 0.$$

This implies

$$\|x\|_{\min} = \text{Sup} \left\{\|\Pi_{\omega}(x)\|: \omega = \omega_1 \otimes \omega_2, \omega_1 \in \sigma(A_1), \omega_2 \in \sigma(A_2)\right\} = 0.$$

Thus we have $x = 0$. Q.E.D.

Theorem 1. Let $A_1$ and $A_2$ be $C^*$-algebras and $T_1$ and $T_2$ be densely defined preclosed operators on $A_1$ and $A_2$. Then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$.

Proof. From Lemma 1, as $T_1$ and $T_2$ are densely defined preclosed operators on $A_1$ and $A_2$, it follows that $\mathcal{D}(T_1^*)$ is $\sigma(A_1^*, A_1)$-dense in $A_1^*$ and $\mathcal{D}(T_2^*)$ is $\sigma(A_2^*, A_2)$-dense in $A_2^*$.

It is easy to verify $\mathcal{D}(T_1^*) \otimes \mathcal{D}(T_2^*)$ is $\sigma((A_1 \otimes_{\min} A_2)^*, (A_1 \otimes_{\min} A))$-dense in $A_1^* \otimes A_2^*$.
By Lemma 2, we can conclude

$$\mathcal{V}(T^*_1) \otimes \mathcal{V}(T^*_2)$$

is a((A_1 \otimes \min A_2)^* \cdot (A_1 \otimes \min A_2))-dense in (A_1 \otimes \min A_2)^*.

Since

$$\mathcal{V}(T^*_1) \otimes \mathcal{V}(T^*_2) \subseteq \mathcal{V}((T_1 \otimes T_2)^*),$$

$$\mathcal{V}((T_1 \otimes T_2)^*)$$

is a((A_1 \otimes \min A_2)^* \cdot (A_1 \otimes \min A_2))-dense in (A_1 \otimes \min A_2)^*. By Lemma 1, $T_1 \otimes T_2$ is preclosed in (A_1 \otimes \min A_2). Q.E.D.

**Corollary 1.** Let $T_1$, $T_2$ be bounded operators on $A_1$ and $A_2$, respectively. Then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes \min A_2$.

Now we turn to studying tensor products of Banach spaces. We will use the notations in [7].

We assume that $E_1$ and $E_2$ denote any two Banach spaces while $E_1^*$ and $E_2^*$ stand for their conjugate spaces. $E_1 \otimes E_2$ and $E_1^* \otimes E_2^*$ denote algebraic tensor products of $E_1$, $E_2$ and $E_1^*$, $E_2^*$ respectively.

If $\beta$ is a norm in $E_1 \otimes E_2$, then $\beta$ induces naturally a norm on $E_1^* \otimes E_2^*$:

$$\| f \|_{\beta^*} = \sup \{ |\langle x, f \rangle| : x \in E_1 \otimes E_2, \| x \|_{\beta} \leq 1 \},$$

where $\langle x, f \rangle$ means, of course, the value

$$\langle x, f \rangle = \sum_{j=1}^{m} \sum_{i=1}^{n} \langle x_{1,j}, f_{1,i} \rangle \langle x_{2,j}, f_{2,i} \rangle,$$

for each $x = \sum_{j=1}^{m} x_{1,j} \otimes x_{2,j} \in E_1 \otimes E_2$.

In the same way, we can define $\beta^{**}$.

If $\beta^{**} = \beta$ on $E_1 \otimes E_2$, we call $\beta$ reflexive.

The completion of $E_1 \otimes E_2$ and $E_1^* \otimes E_2^*$ under $\beta$ and $\beta^*$ are denoted by $E_1 \otimes_{H} E_2$ and $E_1^* \otimes_{H} E_2^*$.

We suppose $\lambda$ is the least norm on $E_1 \otimes E_2$ [7].

**Theorem 2.** If $\beta$ is a reflexive norm on $E_1 \otimes E_2$, $\beta \geq \lambda$ and $T_1$, $T_2$ are densely defined preclosed operators on $E_1$ and $E_2$ respectively, then $T_1 \otimes T_2$ is preclosed on $E_1 \otimes_{H} E_2$.

**Proof.** As $T_1$ and $T_2$ are preclosed, by Lemma 1, $\mathcal{V}(T^*_1)$ is $\sigma(E_1^*, E_1)$-dense in $E_1^*$ and $\mathcal{V}(T^*_2)$ is $\sigma(E_2^*, E_2)$-dense in $E_2^*$ respectively.

Now we prove $E_1^* \otimes_{H} E_2^*$ is $\sigma((E_1 \otimes_{H} E_2)^*, (E_1 \otimes_{H} E_2))^*$-dense in $(E_1 \otimes_{H} E_2)^*$.

Equivalently, we have to prove $E_1^* \otimes_{H} E_2^*$ is total in $(E_1 \otimes_{H} E_2)^*$. Since $\beta \geq \lambda$, from [4] $E_1^* \otimes E_2^* \subseteq (E_1 \otimes_{H} E_2)^*$ and $E_1^* \otimes_{H} E_2^* \subseteq (E_1 \otimes_{H} E_2)^*$. As $\beta$ is reflexive, by Lemma 4.1 of [5], $(E_1 \otimes_{H} E_2) \subseteq (E_1^* \otimes_{H} E_2^*)$.

Let $x$ be an element in $E_1 \otimes_{H} E_2$ such that $f(x) = 0$ for all $f \in E_1^* \otimes_{H} E_2^*$. By continuity of $\beta^*$ and definition, $\beta^{**}(x)$ is the least positive number for which $|f(x)| \leq C\beta^*(f)$, for all $f \in E_1^* \otimes_{H} E_2^*$.

Therefore, $\beta^{**}(x) = 0$.

According to assumption, $\beta(x) = \beta^{**}(x) = 0$ and $x = 0$. 

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Hence, $E_1^* \otimes E_2^*$ is $\sigma((E_1 \otimes_{\beta} E_2)^*, (E_1 \otimes_{\beta} E_2))$-dense in $(E_1 \otimes_{\beta} E_2)^*$. Therefore, $\mathcal{D}(T_1^*) \otimes \mathcal{D}(T_2^*)$ is $\sigma((E_1 \otimes_{\beta} E_2)^*, E_1 \otimes_{\beta} E_2)$-dense in $(E_1 \otimes_{\beta} E_2)^*$.

Since

$$\mathcal{D}(T_1^*) \otimes \mathcal{D}(T_2^*) \subseteq \mathcal{D}((T_1 \otimes T_2)^*),$$

$\mathcal{D}((T_1 \otimes T_2)^*)$ is $\sigma((E_1 \otimes_{\beta} E_2)^*, (E_1 \otimes_{\beta} E_2))$-dense in $(E_1 \otimes_{\beta} E_2)^*$.

By Lemma 1, $T_1 \otimes T_2$ is preclosed. Q.E.D.

Now we prove that the injective $C^*$-cross norm is reflexive. Using this property, we can give another proof of Theorem 1.

**Theorem 3.** If $A_1$ and $A_2$ are $C^*$-algebras, the injective $C^*$-cross norm on the algebraic tensor product $A_1 \otimes A_2$ is reflexive.

**Proof.** Let 

$$A = A_1 \otimes A_2, \quad V = A_1^* \otimes A_2^*.$$ 

$S_{a^*}, S_a$ denote the unit ball of $V, A$ respectively, that is,

$$S_{a^*} = \{\omega : \|\omega\|_{a^*} \leq 1, \omega \in V\},$$
$$S_a = \{a : \|a\|_a \leq 1, a \in A\}.$$

We further set

$$\langle x, a\omega \rangle = \langle ax, \omega \rangle, \quad \langle x, a\omega \rangle = \langle xa, \omega \rangle.$$ 

Since $\|a\omega\|_{a^*} \leq \|a\|_a \|\omega\|_{a^*}$ for $a \in A_1 \otimes A_2$ and $\omega \in V$, $V$ is invariant under $A$.

That is, if $\omega \in V$, then $a\omega \in V$, $a\omega \in V$ for all $a \in A$.

For $a \in A$, we define $\|a\|_a = \text{Sup}\{\|a\omega\|_{a^*} : \omega \in V, \|\omega\|_{a^*} \leq 1\}$.

It is easy to verify

$$\|a + b\|_a \leq \|a\|_a + \|b\|_a, \quad \|ab\|_a \leq \|a\|_a \|b\|_a, \quad \|\lambda a\|_a = |\lambda| \|a\|_a, \quad \|a\|_a \leq \|a\|_{a^*} \text{ for } a, b \in A.$$

According to the minimal property of the norm [1], we have

$$\|a\|_a = \|a\|_{a^*}.$$ 

$$\|a\|_a = \text{Sup}\{\|a\omega\|_{a^*} : \omega \in S_{a^*}\}$$
$$= \text{Sup}\{\|b, a\omega\| : b \in S_a, \omega \in S_{a^*}\}$$
$$= \text{Sup}\{\|ba, \omega\| : b \in S_a, \omega \in S_{a^*}\}$$
$$= \text{Sup}\{\|a, \omega b\| : b \in S_a, \omega \in S_{a^*}\}$$
$$\leq \text{Sup}\{\|a, \omega\| : \omega \in S_{a^*}\} \leq \|a\|_{a^*}.$$ 

Therefore, $\|a\|_a = \|a\|_{a^*}$. Q.E.D.

**Corollary 2.** Let $A_1$ and $A_2$ be $C^*$-algebras and $T_1$ and $T_2$ be densely defined preclosed operators on $A_1$ and $A_2$. Then $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$.

**Proof.** Since $\|\cdot\|_{\min}$ is reflexive and $\lambda \leq \|\cdot\|_{\min}$, by Theorem 2, $T_1 \otimes T_2$ is preclosed on $A_1 \otimes_{\min} A_2$. Q.E.D.

In fact, Corollary 2 gives another proof of Theorem 1.
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REFERENCES


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