ON THE EXTENSION OF $H^p$-FUNCTIONS IN POLYDISCS

P. S. CHEE

Abstract. For $N = 2$ or $3$ it is shown that if $E$ is the zero set of a holomorphic function in $U^N$ satisfying the separation condition of Alexander [1], viz., there exist $r \in (0,1)$ and $\delta > 0$ such that $|\alpha - \beta| \geq \delta$ whenever $(z', \alpha, z'') \neq (z', \beta, z'')$ are both in $(Q^{k-1} \times U \times Q^{N-k}) \cap E$, where $Q = \{ \lambda \in \mathbb{C}: r < |\lambda| < 1 \}$, then (a) $E$ is the zero set of some $F \in H^\infty(U^N)$, and (b) $0 < p \leq \infty$, every $g \in H^p(E)$ such that $|g|^p$ has a pluriharmonic majorant on $E$ extends to $G \in H^p(U^N)$. This generalizes earlier results of the author [3] and Zarantonello [9].

I. Introduction. For $r, s > 0$ and a positive integer $N$, let $U(r) = \{ \lambda \in \mathbb{C}: |\lambda| < r \}$, $Q(r, s) = \{ \lambda \in \mathbb{C}: r < |\lambda| < s \}$, $U^N(r) = U(r) \times \cdots \times U(r)$ and $Q^N(r, s) = Q(r, s) \times \cdots \times Q(r, s)$ ($N$ copies). As usual, we write $U$ for $U(1)$ and $U^N$ for $U^N(1)$, the open unit polydisc.

For any domain $\Omega$ in $\mathbb{C}^N$, $H(\Omega)$ denotes the set of all holomorphic functions in $\Omega$, and $H^p(\Omega)$ the set of all bounded ones. If $\Omega$ is a polydomain in $\mathbb{C}^N$, i.e. a Cartesian product of $N$ open connected subsets of $\mathbb{C}$, and $0 < p < \infty$, let $H^p(\Omega)$ denote the set of all $f \in H(\Omega)$ such that $|f|^p$ has an $n$-harmonic majorant in $\Omega$. If $f \in H(\Omega)$, then $Z(f) = f^{-1}(0)$ denotes its zero set. If $E = Z(f)$ for some $f \in H(U^N)$, let $H^p(E)$ denote the set of all $f \in H(E)$ such that $|f(z)|^p \leq u(z)$, $z \in E$, for some pluriharmonic function $u$ on $E$. The set of all invertible elements of an algebra $A$ is denoted by $\text{Inv} A$.

Now let $N > 1$ and $E = Z(f)$ for some $f \in H(U^N)$. If $g \in H(E)$, then by Cartan's Theorem B, there exists a $G \in H(U^N)$ such that $G = g$ on $E$ (see [4, p. 245]). Here we consider the problem of finding extensions $G \in H^p(U^N)$ when $g \in H^p(E)$. Without further conditions on $E$ or on $g$, there are in general no $H^p$ extensions (see [1]).

In [1], Alexander shows that if $g \in H^\infty(E)$, then there exists an extension $G \in H^\infty(U^N)$ if $E$ satisfies the following conditions:

(A): There exist $r \in (0,1)$ and $\delta > 0$ such that $|\alpha - \beta| \geq \delta$ whenever $1 \leq k \leq N$ and $(z', \alpha, z'') \neq (z', \beta, z'')$ are both in $(Q^{k-1} \times U \times Q^{N-k}) \cap E$, where $Q = Q(r, 1)$, and

(R): $\text{dist}(E, Q^N) > 0$.

The condition (R), due to Rudin, implies that $E = Z(F)$ for some $F \in H^\infty(U^N)$ with $F^{-1}$ bounded in $Q^N$ (see [5, Theorem 4.8.3]).
In [3] it was shown that Alexander’s result holds if (R) is replaced by the condition $(Z)_N$: There exists a continuous function $\eta: [r, 1) \rightarrow [r, 1)$ such that

\begin{equation}
|z_N| \leq \eta(|z_1| + \cdots + |z_{N-1}|)/(N-1)
\end{equation}

whenever $z = (z_1, \ldots, z_N) \in Q^N \cap E$.

This condition, introduced in [8], also implies that $E = Z(F)$ for some $F \in H^\infty(U^N)$ (see [3]).

In [9] Zarantonello showed that if $E$ satisfies (A) and $(Z)_N$, $0 < p < \infty$, then every $g \in H^p(E)$ has an extension $G \in H^p(U^N)$.

The purpose of this article is to show that for $N = 2$ or $3$, (A) alone is sufficient for the above results. In fact we shall prove

**Theorem 1.1.** If $N = 2$ or $3$, $E = Z(f)$ for some $f \in H(U^N)$, and $E$ satisfies (A), then

(a) $E = Z(F)$ for some $F \in H^\infty(U^N)$, and
(b) for $0 < p \leq \infty$, every $g \in H^p(E)$ extends to a $G \in H^p(U^N)$.

**II. A zero set for bounded holomorphic functions.** Suppose $E = Z(f)$ for some $f \in H(U^N)$ and satisfies condition (A) of §I. We may choose $f$ so that it generates the ideal sheaf of $E$ (see [4, p. 251]). We show first that $E = Z(F)$ for some $F \in H^\infty(U^N)$, if $N \leq 3$.

Consider the case $k = N$ in (A). Write $z = (z', z_N)$, where $z' = (z_1, \ldots, z_{N-1}) \in C^{N-1}$. Then (A) implies that for each $z' \in Q^{N-1}$, $f(z', \cdot)$ has only a finite number $m = m(z')$ of zeros in $U$. Let $\gamma = \gamma(z')$ be a circle in $U$ with centre 0 and enclosing all these zeros. Then

\begin{equation}
m = \frac{1}{2\pi i} \int_{\gamma} \frac{D_N f(z', \lambda)}{f(z', \lambda)} d\lambda.
\end{equation}

Note that $m$ is independent of the choice of $\gamma$ as long as the radius $s$ of $\gamma$ is sufficiently close to 1. Since $f(z', \lambda) \neq 0$ for all $s \leq |\lambda| < 1$, the continuity of $f$ implies that there exists a neighborhood $W' \times W''$ of $\{z'\} \times \{\lambda \in C: s \leq |\lambda| < 1\}$ in which $f \neq 0$. It follows from (2.1) that $m$ is continuous in $W'$. Hence $m$ is continuous in $Q^{N-1}$. Since $m$ is integer-valued and $Q^{N-1}$ is connected, $m$ is constant in $Q^{N-1}$.

By [1, p. 486], $D_N f \neq 0$ in $(Q^{N-1} \times U) \cap E$. It follows that $(Q^{N-1} \times U) \cap E$ is an unbranched analytic cover of $Q^{N-1}$ of $m$ sheets (see [6]). Hence there exist $m$ locally defined holomorphic functions $\alpha_1, \ldots, \alpha_m$ on $Q^{N-1}$ such that

\begin{equation}
(Q^{N-1} \times U) \cap E = \{(z', z_N) \in Q^{N-1} \times U: z_N = \alpha_j(z'), \text{ for some } 1 \leq j \leq m\}.
\end{equation}

Let $(\rho_k, \theta_k)$ be the polar coordinates of $z_k$, $\rho = (\rho_1, \ldots, \rho_{N-1})$, $\theta = (\theta_1, \ldots, \theta_{N-1})$. Then $\alpha_j(\rho, \theta) = \alpha_j(z')$ is continuous on $(r, 1)^{N-1} \times [0, 2\pi]^{N-1}$. Let

$$
\alpha(\rho, \theta) = \max\{|\alpha_j(\rho, \theta)|: 1 \leq j \leq m\},
$$

$$
\eta(\rho) = \max\{\alpha(\rho, \theta): \theta \in [0, 2\pi]^{N-1}\}.
$$
Clearly, $\alpha$ is continuous on $(r, l)^{N-1} \times [0, 2\pi]^{N-1}$. For each $r_0 \in (r, 1)^{N-1}$, choose an open interval $I$ such that $r_0 \in I$ and $I \subseteq (r, 1)^{N-1}$. Then the uniform continuity of $\alpha$ on $I \times [0, 2\pi]^{N-1}$ implies that $\eta$ is continuous on $I$. Hence $\eta$ is continuous on $(r, 1)^{N-1}$. Since $|\alpha_j(\rho, \theta)| \neq 1$ for all $1 \leq j \leq m$, $\theta \in [0, 2\pi]^{N-1}$, $\eta(\rho) < 1$. By increasing $r$ slightly if necessary, we see that there exists a continuous function $\eta_N$: $(r, 1)^{N-1} \to [0, 1)$ such that $|z_N| \leq \eta_N(|z_1|, \ldots, |z_{N-1}|)$ whenever $(z', z_N) \in (Q^{N-1} \times U) \cap E$.

Next, for $1 \leq k \leq N$, let $W_k = Q^{k-1} \times U \times Q^{N-k}$. For $(z', z_N) \in W_N$, define

$$F_N(z) = \prod_{j=1}^{m} (z_N - \alpha_j(z')),$$

where $\alpha_1, \ldots, \alpha_m$ are as in (2.2). Then $F_N \in H^\infty(W_N)$ and $fF_N^{-1} \in \text{Inv} H(W_N)$. (See [8, p. 312].)

Similarly, by considering other values of $k$ instead of $N$, we see that (A) implies the existence of continuous functions $\eta_k$ analogous to $\eta_N$ and functions $F_k$ analogous to $F_N$.

It is convenient to introduce the following conditions:

$(Z')_k$: There exist $r \in (0, 1)$ and a continuous function $\xi: [r, 1)^{N-1} \to [0, 1)$ such that

$$|z_k| \leq \xi(|z_1|, \ldots, |z_{k-1}|, |z_{k+1}|, \ldots, |z_N|)$$

whenever $z \in W_k \cap E$. We say that $E$ satisfies $(Z')$ if it satisfies $(Z')_k$ for $1 \leq k \leq N$.

From the preceding discussion, by increasing $r$ if necessary and by taking $\xi = \max(\eta_k: 1 \leq k \leq N)$, we have the following:

**Lemma 2.1.** If $E = Z(f)$ satisfies (A), then

(a) $E$ satisfies $(Z')$;

(b) for $1 \leq k \leq N$, there exists $F_k \in H^\infty(W_k)$ such that $fF_k^{-1} \in \text{Inv} H(W_k)$.

**Remark 2.2.** (i) If $N = 2$, then $(Z')_2 = (Z)_2$, and Theorem 1.1 follows from [3, Theorems 3.1 and 4.1] and [9, Theorem 6].

(ii) If $E = Z(f)$ satisfies $(Z')_N$, then for each $z' \in Q^{N-1}$, $f(z', \cdot)$ has only a finite number $m = m(z')$ of zeros in $U$. The argument above shows that $m$ is constant on $Q^{N-1}$ and $F_N$, defined by (2.3), where the $\alpha_j$'s are the zeros of $f(z', \cdot)$, is in $H^\infty(W_N)$ (see [8, p. 312]). Similarly for other $k$ in place of $N$. Hence (b) is a consequence of (a) in Lemma 2.1.

The rest of the paper is devoted to the case $N = 3$ of Theorem 1.1.

**Theorem 2.3.** If $N = 3$ and $E = Z(f)$ satisfies $(Z')$ then there exists an $F \in H^\infty(U^3)$ such that

(a) $fF^{-1} \in \text{Inv} H(U^3)$,

(b) $FF_3^{-1} \in \text{Inv} H^\infty(W_3)$.

**Proof.** Fix $s \in (r, 1)$. Let

$$V_1 = U(s) \times U^2, \quad V_2 = U \times U(s) \times U, \quad V_3 = Q^2 \times U = W_3.$$

Consider the polydisc $V_1$. 

Let $\xi$ be as in $(Z)_3$. Then for each $t \in [r, 1)$, $\tilde{\eta}(t) = \max\{\xi(\tau, t) : \tau \in [r, s]\}$ is continuous and $\tilde{\eta}(t) < 1$. Hence if we replace $\eta$ by $\tilde{\eta}$ in $(Z)_3$, then the proof in [3, §III] gives $f_1 \in H^\infty(V_1)$ such that
\begin{align}
(2.5) & \quad ff_1^{-1} \in \text{Inv} H(V_1), \\
(2.6) & \quad f_1F_3^{-1} \in \text{Inv} H^\infty(W_3 \cap V_1).
\end{align}

Similarly, for the polydisc $V_2$, there exists $f_2 \in H^\infty(V_2)$ such that
\begin{align}
(2.7) & \quad ff_2^{-1} \in \text{Inv} H(V_2), \\
(2.8) & \quad f_2F_3^{-1} \in \text{Inv} H^\infty(W_3 \cap V_2).
\end{align}

Let $f_3 = F_3 \in H^\infty(V_3)$. Since $f_1f_2^{-1} = f_1^{-1} : ff_2^{-1}$, (2.5) and (2.7) imply that
\[ f_1f_2^{-1} \in \text{Inv} H(V_1 \cap V_2). \]

Since $f_1f_2^{-1} = f_1^{-1} : F_3f_2^{-1}$, (2.6) and (2.8) imply that
\[ f_1f_2^{-1} \in \text{Inv} H^\infty(W_3 \cap V_1 \cap V_2). \]

Since the distinguished boundary of $V_1 \cap V_2$ is contained in that of $W_3 \cap V_1 \cap V_2$, it follows from the maximum modulus theorem that $f_1f_2^{-1} \in \text{Inv} H^\infty(V_1 \cap V_2)$. Together with (2.6) and (2.8), this shows that $\{f_1, f_2, f_3\}$ forms a set of bounded Cousin data for the cover $(V_1, V_2, V_3)$ of $U^3$. By Stout’s theorem [7], there exists an $F \in H^\infty(U^3)$ satisfying (a) and (b).

Theorem 1.1(a) follows from Lemma 2.1 and Theorem 2.3.

**III. Extension of $H^p$-functions.** We now prove part (b) of Theorem 1.1 for $N = 3$. Let $V_1, V_2, V_3$ be as in the proof of Theorem 2.3, $0 < p \leq \infty$, and $g \in H^p(E)$. Let $\alpha_1, \ldots, \alpha_m$ be as in (2.2). Define
\[ g_3(z) = \sum_{i=1}^m g(\alpha_i(z')) \prod_{\substack{j \neq i \leq m \atop 1 \leq j \leq m}} \frac{z_j - \alpha_j(z')}{\alpha_i(z') - \alpha_j(z')}, \quad z = (z', z_3) \in V_3. \]

Then $g_3 \in H^p(V_3)$ and $g_3 = g$ on $V_3 \cap E$ (see [9, p. 522] and [3, p. 112]).

Let $s' \in (s, 1)$, $V'_1 = U(s') \times U^2$, and
\begin{align*}
\Omega_1 &= U(s') \times U^2, \quad \Omega_2 = U^2(s') \times U, \quad \Omega_3 = Q(r, s') \times Q(r, 1) \times U.
\end{align*}

Then $\{\Omega_i : 1 \leq i \leq 3\}$ is an open cover of the polydisc $V'_1$. With the function $\tilde{\eta}(t) = \max\{\xi(\tau, t) : \tau \in [r, s']\}$ in place of $\eta$ in $(Z)_3$, the proofs of [3, §IV] and [9] show that there exists $g_1 \in H^p(V'_1)$ such that $g_1 = g$ on $V'_1 \cap E$. The construction of $g_1$ (see [9, p. 524]) gives
\[ g_1 = g_3 + f_3F \]
in $\Omega_1 \cap \Omega_3 = Q(r, s') \times Q(r, 1) \times U = V_1 \cap V_3$, where $F$ is as in Theorem 2.3 and $f_3 \in H^p(V_1 \cap V_3)$. Since $V'_1 \supset V_1$, we have $g_1 \in H^p(V'_1)$, $g_1 = g$ on $V_1 \cap E$ and satisfies (3.1).
Similarly, there exists $g_2 \in H^p(V_2)$ such that $g_2 = g$ on $V_2 \cap E$, and
\begin{equation}
(3.2) \quad g_2 = g_3 + f_{23}F \quad \text{in } V_2 \cap V_3
\end{equation}
where $f_{23} \in H^p(V_2 \cap V_3)$.

Now let $f_{12} = (g_1 - g_2)/F$ in $V_1 \cap V_2$. Since $g_1 - g_2 = 0$ on $V_1 \cap V_2 \cap E$, and $F$ generates the ideal sheaf of $E$, $f_{12} \in H(V_1 \cap V_2)$.

By the continuity of $\xi$, $c = \max \{\xi(r_1, r_2) : r_1, r_2 \in [r, s]\} < 1$. Choose $c' \in (c, 1)$. Then $F^{-1}$ is bounded in $\Omega = Q^2(r, s) \times Q(c', 1)$. By Theorem 2.3(b), $F = F_3\psi$, where $\psi \in \text{Inv } H^{\infty}(W_3)$. Hence $F^{-1}$ is bounded in $\Omega$. Since the distinguished boundary of $V_1 \cap V_2$ is contained in $\bar{\Omega}$, it follows that $f_{12} \in H^p(V_1 \cap V_2)$.

We now appeal to the theorems of Andreotti and Stoll [2] (for $p = \infty$) and Zarantonello [10, p. 493] (for $0 < p < \infty$) to conclude that $g$ has an extension $G \in H^p(U^3)$.

ACKNOWLEDGEMENT. The author would like to thank Professor Abdus Salam, the International Atomic Energy Agency and UNESCO for partial support and for hospitality at the International Centre for Theoretical Physics, Trieste, where this work was done. He also thanks the referee for useful comments.

REFERENCES


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MALAYA, KUALA LUMPUR, MALAYSIA