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ON THE EXTENSION OF H^p-FUNCTIONS IN POLYDISCS

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ABSTRACT. For $N = 2$ or 3 it is shown that if E is the zero set of a holomorphic function in U^N satisfying the separation condition of Alexander [1], viz., there exist $r \in (0,1)$ and $\delta > 0$ such that $|\alpha - \beta| \geq \delta$ whenever $(z', \alpha, z'') \neq (z', \beta, z'')$ are both in $(Q^{k-1} \times U \times Q^{N-k}) \cap E$, where $Q = \{\lambda \in \mathbf{C}: r < |\lambda| < 1\}$, then (a) E is the zero set of some $F \in H^\infty(U^N)$, and (b) $0 < p \leq \infty$, every $g \in H(E)$ such that $|g|^p$ has a pluriharmonic majorant on E extends to a $G \in H^p(U^N)$. This generalizes earlier results of the author [3] and Zarantonello [9].

I. Introduction. For $r, s > 0$ and a positive integer N, let $U(r) = \{\lambda \in \mathbf{C}: |\lambda| < r\}$, $Q(r,s) = \{\lambda \in \mathbf{C}: r < |\lambda| < s\}$, $U^N(r) = U(r) \times \cdots \times U(r)$ and $Q^N(r,s) = Q(r,s) \times \cdots \times Q(r,s)$ (N copies). As usual, we write U for $U(1)$ and U^N for $U^N(1)$, the open unit polydisc.

For any domain Ω in $\mathbf{C}^N$, $H(\Omega)$ denotes the set of all holomorphic functions in Ω, and $H^\infty(\Omega)$ the set of all bounded ones. If Ω is a polydomain in $\mathbf{C}^N$, i.e. a Cartesian product of N open connected subsets of $\mathbf{C}$, and $0 < p < \infty$, let $H^p(\Omega)$ denote the set of all $f \in H(\Omega)$ such that $|f|^p$ has an n-harmonic majorant in Ω. If $f \in H(\Omega)$, then $Z(f) = f^{-1}(0)$ denotes its zero set. If $E = Z(f)$ for some $f \in H(U^N)$, let $H^p(E)$ denote the set of all $f \in H(E)$ such that $|f(z)|^p \leq u(z)$, $z \in E$, for some pluriharmonic function u on E. The set of all invertible elements of an algebra A is denoted by $\operatorname{Inv} A$.

Now let $N > 1$ and $E = Z(f)$ for some $f \in H(U^N)$. If $g \in H(E)$, then by Cartan's Theorem B, there exists a $G \in H(U^N)$ such that $G = g$ on E (see [4, p. 245]). Here we consider the problem of finding extensions $G \in H^p(U^N)$ when $g \in H^p(E)$. Without further conditions on E or on g, there are in general no H^p extensions (see [1]).

In [1], Alexander shows that if $g \in H^\infty(E)$, then there exists an extension $G \in H^\infty(U^N)$ if E satisfies the following conditions:

(A): There exist $r \in (0,1)$ and $\delta > 0$ such that $|\alpha - \beta| \geq \delta$ whenever $1 \leq k \leq N$ and $(z', \alpha, z'') \neq (z', \beta, z'')$ are both in $(Q^{k-1} \times U \times Q^{N-k}) \cap E$, where $Q = Q(r, 1)$, and

(R): $\operatorname{dist}(E, Q^N) > 0$.

The condition (R), due to Rudin, implies that $E = Z(F)$ for some $F \in H^\infty(U^N)$ with F^{-1} bounded in Q^N (see [5, Theorem 4.8.3]).

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In [3] it was shown that Alexander's result holds if (R) is replaced by the condition $(\mathrm{Z})_N$: There exists a continuous function $\eta\colon [r,1) \to [r,1)$ such that

$$(1.1) \qquad |z_N| \leqslant \eta((|z_1| + \cdots + |z_{N-1}|)/(N-1)) \quad \text{whenever } z = (z_1,\ldots,z_N) \in Q^N \cap E.$$

This condition, introduced in [8], also implies that $E = Z(F)$ for some $F \in H^\infty(U^N)$ (see [3]).

In [9] Zarantonello showed that if E satisfies (A) and $(\mathrm{Z})_N$, $0 < p < \infty$, then every $g \in H^p(E)$ has an extension $G \in H^p(U^N)$.

The purpose of this article is to show that for $N = 2$ or 3, (A) alone is sufficient for the above results. In fact we shall prove

THEOREM 1.1. *If $N = 2$ or 3, $E = Z(f)$ for some $f \in H(U^N)$, and E satisfies* (A), *then*

(a) *$E = Z(F)$ for some $F \in H^\infty(U^N)$, and*

(b) *for $0 < p \leqslant \infty$, every $g \in H^p(E)$ extends to a $G \in H^p(U^N)$.*

II. A zero set for bounded holomorphic functions. Suppose $E = Z(f)$ for some $f \in H(U^N)$ and satisfies condition (A) of §I. We may choose f so that it generates the ideal sheaf of E (see [4, p. 251]). We show first that $E = Z(F)$ for some $F \in H^\infty(U^N)$, if $N \leqslant 3$.

Consider the case $k = N$ in (A). Write $z = (z', z_N)$, where $z' = (z_1,\ldots,z_{N-1}) \in \mathbf{C}^{N-1}$. Then (A) implies that for each $z' \in Q^{N-1}$, $f(z', \cdot)$ has only a finite number $m = m(z')$ of zeros in U. Let $\gamma = \gamma(z')$ be a circle in U with centre 0 and enclosing all these zeros. Then

$$(2.1) \qquad m = \frac{1}{2\pi i}\int_\gamma \frac{D_N f(z',\lambda)}{f(z',\lambda)}\,d\lambda.$$

Note that m is independent of the choice of γ as long as the radius s of γ is sufficiently close to 1. Since $f(z',\lambda) \neq 0$ for all $s \leqslant |\lambda| < 1$, the continuity of f implies that there exists a neighborhood $W' \times W''$ of $\{z'\} \times \{\lambda \in \mathbf{C}\colon s \leqslant \{\lambda\} < 1\}$ in which $f \neq 0$. It follows from (2.1) that m is continuous in W'. Hence m is continuous in Q^{N-1}. Since m is integer-valued and Q^{N-1} is connected, m is constant in Q^{N-1}.

By [1, p. 486], $D_N f \neq 0$ in $(Q^{N-1} \times U) \cap E$. It follows that $(Q^{n-1} \times U) \cap E$ is an unbranched analytic cover of Q^{N-1} of m sheets (see [6]). Hence there exist m locally defined holomorphic functions $\alpha_1,\ldots,\alpha_m$ on Q^{N-1} such that

$$(2.2) \qquad (Q^{N-1} \times U) \cap E = \{(z', z_N) \in Q^{N-1} \times U\colon z_N = \alpha_j(z'), \text{ for some } 1 \leqslant j \leqslant m\}.$$

Let (ρ_k, θ_k) be the polar coordinates of z_k, $\rho = (\rho_1,\ldots,\rho_{N-1})$, $\theta = (\theta_1,\ldots,\theta_{N-1})$. Then $\alpha_j(\rho,\theta) = \alpha_j(z')$ is continuous on $(r,1)^{N-1} \times [0,2\pi]^{N-1}$. Let

$$\alpha(\rho,\theta) = \max\{|\alpha_j(\rho,\theta)|: 1 \leqslant j \leqslant m\},$$

$$\eta(\rho) = \max\{\alpha(\rho,\theta)\colon \theta \in [0,2\pi]^{N-1}\}.$$

Clearly, α is continuous on $(r,1)^{N-1}\times[0,2\pi]^{N-1}$. For each $r_0\in(r,1)^{N-1}$, choose an open interval I such that $r_0\in I$ and $\bar{I}\subseteq(r,1)^{N-1}$. Then the uniform continuity of α on $\bar{I}\times[0,2\pi]^{N-1}$ implies that η is continuous on $\bar{I}$. Hence η is continuous on $(r,1)^{N-1}$. Since $|\alpha_j(\rho,\theta)|\neq 1$ for all $1\leqslant j\leqslant m$, $\theta\in[0,2\pi]^{N-1}$, $\eta(\rho)<1$. By increasing r slightly if necessary, we see that there exists a continuous function η_N: $[r,1)^{N-1}\to[0,1)$ such that $|z_N|\leqslant\eta_N(|z_1|,\ldots,|z_{N-1}|)$ whenever $(z',z_N)\in(Q^{N-1}\times U)\cap E$.

Next, for $1\leqslant k\leqslant N$, let $W_k=Q^{k-1}\times U\times Q^{N-k}$. For $(z',z_N)\in W_N$, define

$$F_N(z)=\prod_{j=1}^{m}\left(z_N-\alpha_j(z')\right), \tag{2.3}$$

where $\alpha_1,\ldots,\alpha_m$ are as in (2.2). Then $F_N\in H^\infty(W_N)$ and $fF_N^{-1}\in\operatorname{Inv}H(W_N)$. (See [**8**, p. 312].)

Similarly, by considering other values of k instead of N, we see that (A) implies the existence of continuous functions η_k analogous to η_N and functions F_k analogous to F_N.

It is convenient to introduce the following conditions:

$(\mathrm{Z}')_k$: There exist $r\in(0,1)$ and a continuous function ξ: $[r,1)^{N-1}\to[0,1)$ such that

$$|z_k|\leqslant\xi(|z_1|,\ldots,|z_{k-1}|,|z_{k+1}|,\ldots,|z_N|) \tag{2.4}$$

whenever $z\in W_k\cap E$. We say that E satisfies (Z') if it satisfies $(\mathrm{Z}')_k$ for $1\leqslant k\leqslant N$.

From the preceding discussion, by increasing r if necessary and by taking $\xi=\max\{\eta_k\colon 1\leqslant k\leqslant N\}$, we have the following:

LEMMA 2.1. *If $E=Z(f)$ satisfies* (A), *then*

(a) *E satisfies* (Z');

(b) *for $1\leqslant k\leqslant N$, there exists $F_k\in H^\infty(W_k)$ such that $fF_k^{-1}\in\operatorname{Inv}H(W_k)$.*

REMARK 2.2. (i) If $N=2$, then $(\mathrm{Z}')_2=(\mathrm{Z})_2$, and Theorem 1.1 follows from [**3**, Theorems 3.1 and 4.1] and [**9**, Theorem 6].

(ii) If $E=Z(f)$ satisfies $(\mathrm{Z}')_N$, then for each $z'\in Q^{N-1}$, $f(z',\cdot)$ has only a finite number $m=m(z')$ of zeros in U. The argument above shows that m is constant on Q^{N-1} and F_N, defined by (2.3), where the α_j's are the zeros of $f(z',\cdot)$, is in $H^\infty(W_N)$ (see [**8**, p. 312]). Similarly for other k in place of N. Hence (b) is a consequence of (a) in Lemma 2.1.

The rest of the paper is devoted to the case $N=3$ of Theorem 1.1.

THEOREM 2.3. *If $N=3$ and $E=Z(f)$ satisfies* (Z') *then there exists an $F\in H^\infty(U^3)$ such that*

(a) *$fF^{-1}\in\operatorname{Inv}H(U^3)$,*

(b) *$FF_3^{-1}\in\operatorname{Inv}H^\infty(W_3)$.*

PROOF. Fix $s\in(r,1)$. Let

$$V_1=U(s)\times U^2,\quad V_2=U\times U(s)\times U,\quad V_3=Q^2\times U=W_3.$$

Consider the polydisc V_1.

Let ξ be as in $(Z')_3$. Then for each $t \in [r, 1)$, $\tilde{\eta}(t) = \max\{\xi(\tau, t): \tau \in [r, s]\}$ is continuous and $\tilde{\eta}(t) < 1$. Hence if we replace η by $\tilde{\eta}$ in $(Z)_3$, then the proof in [3], §III] gives $f_1 \in H^\infty(V_1)$ such that

$$ff_1^{-1} \in \operatorname{Inv} H(V_1), \tag{2.5}$$

$$f_1 F_3^{-1} \in \operatorname{Inv} H^\infty(W_3 \cap V_1). \tag{2.6}$$

Similarly, for the polydisc V_2, there exists $f_2 \in H^\infty(V_2)$ such that

$$ff_2^{-1} \in \operatorname{Inv} H(V_2), \tag{2.7}$$

$$f_2 F_3^{-1} \in \operatorname{Inv} H^\infty(W_3 \cap V_2). \tag{2.8}$$

Let $f_3 = F_3 \in H^\infty(V_3)$.

Since $f_1 f_2^{-1} = f_1 f^{-1} \cdot ff_2^{-1}$, (2.5) and (2.7) imply that

$$f_1 f_2^{-1} \in \operatorname{Inv} H(V_1 \cap V_2).$$

Since $f_1 f_2^{-1} = f_1 F_3^{-1} \cdot F_3 f_2^{-1}$, (2.6) and (2.8) imply that

$$f_1 f_2^{-1} \in \operatorname{Inv} H^\infty(W_3 \cap V_1 \cap V_2).$$

Since the distinguished boundary of $V_1 \cap V_2$ is contained in that of $W_3 \cap V_1 \cap V_2$, it follows from the maximum modulus theorem that $f_1 f_2^{-1} \in \operatorname{Inv} H^\infty(V_1 \cap V_2)$. Together with (2.6) and (2.8), this shows that $\{f_1, f_2, f_3\}$ forms a set of bounded Cousin data for the cover $\{V_1, V_2, V_3\}$ of U^3. By Stout's theorem [7], there exists an $F \in H^\infty(U^3)$ satisfying (a) and (b).

Theorem 1.1(a) follows from Lemma 2.1 and Theorem 2.3.

III. Extension of H^p-functions. We now prove part (b) of Theorem 1.1 for $N = 3$. Let V_1, V_2, V_3 be as in the proof of Theorem 2.3, $0 < p \leqslant \infty$, and $g \in H^p(E)$. Let $\alpha_1, \ldots, \alpha_m$ be as in (2.2). Define

$$g_3(z) = \sum_{i=1}^{m} g(z', \alpha_i(z')) \prod_{\substack{j \neq i \\ 1 \leqslant j \leqslant m}} \frac{z_3 - \alpha_j(z')}{\alpha_i(z') - \alpha_j(z')}, \qquad z = (z', z_3) \in V_3.$$

Then $g_3 \in H^p(V_3)$ and $g_3 = g$ on $V_3 \cap E$ (see [9, p. 522] and [3, p. 112]).

Let $s' \in (s, 1)$, $V_1' = U(s') \times U^2$, and

$$\Omega_1 = U(s) \times U^2, \quad \Omega_2 = U^2(s) \times U, \quad \Omega_3 = Q(r, s') \times Q(r, 1) \times U.$$

Then $\{\Omega_i\colon 1 \leqslant i \leqslant 3\}$ is an open cover of the polydisc V_1'. With the function $\tilde{\eta}(t) = \max\{\xi(\tau, t): \tau \in [r, s']\}$ in place of η in $(Z)_3$, the proofs of [3, §IV] and [9] show that there exists $g_1 \in H^p(V_1')$ such that $g_1 = g$ on $V_1' \cap E$. The construction of g_1 (see [9, p. 524]) gives

$$g_1 = g_3 + f_3 F \tag{3.1}$$

in $\Omega_1 \cap \Omega_3 = Q(r, s) \times Q(r, 1) \times U = V_1 \cap V_3$, where F is as in Theorem 2.3 and $f_{13} \in H^p(V_1 \cap V_3)$. Since $V_1' \supseteq V_1$, we have $g_1 \in H^p(V_1)$, $g_1 = g$ on $V_1 \cap E$ and satisfies (3.1).

Similarly, there exits $g_2 \in H^p(V_2)$ such that $g_2 = g$ on $V_2 \cap E$, and

$$g_2 = g_3 + f_{23}F \quad \text{in } V_2 \cap V_3 \tag{3.2}$$

where $f_{23} \in H^p(V_2 \cap V_3)$.

Now let $f_{12} = (g_1 - g_2)/F$ in $V_1 \cap V_2$. Since $g_1 - g_2 = 0$ on $V_1 \cap V_2 \cap E$, and F generates the ideal sheaf of E, $f_{12} \in H(V_1 \cap V_2)$.

By the continuity of ξ, $c = \max\{\xi(r_1, r_2): r_1, r_2 \in [r, s]\} < 1$. Choose $c' \in (c, 1)$. Then F_3^{-1} is bounded in $\Omega = Q^2(r, s) \times Q(c', 1)$. By Theorem 2.3(b), $F = F_3\psi$, where $\psi \in \operatorname{Inv} H^\infty(W_3)$. Hence F^{-1} is bounded in Ω. Since the distinguished boundary of $V_1 \cap V_2$ is contained in $\overline{\Omega}$, it follows that $f_{12} \in H^p(V_1 \cap V_2)$.

We now appeal to the theorems of Andreotti and Stoll [**2**] (for $p = \infty$) and Zarantonello [**10**, p. 493] (for $0 < p < \infty$) to conclude that g has an extension $G \in H^p(U^3)$.

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