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REMOVABLE SINGULARITY SETS FOR ANALYTIC FUNCTIONS HAVING MODULUS WITH BOUNDED LAPLACE MASS

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ABSTRACT. We prove that certain closed sets are removable singularity sets for analytic functions having modulus with bounded Laplace mass. As a special case, we find that every function which is analytic and with modulus having a harmonic majorant outside an analytic set extends analytically across this set.

1. Introduction. Denote by B^n the unit ball in $\mathbf{C}^n$ and let P be a closed subset of B^n. The purpose of this note is to prove a theorem which gives sufficient conditions on P in order to insure that every analytic function f on $B^n \setminus P$, such that $\Delta |f|^q$ has locally finite mass near P for some $q > 0$, has a (unique) extension to B^n. (Δ denotes the Laplace operator.)

2. A proposition.

PROPOSITION. *If P has vanishing Newton capacity and if $\phi \geq 0$ is subharmonic on $B^n \setminus P$, then the following conditions are equivalent*:

(i) *φ has a harmonic majorant on $B^n \setminus P$*;

(ii) *$\int_{B^n \setminus P} G(X, Y)\Delta\varphi(Y) \not\equiv +\infty$, where G is Green's function for B^n*;

(iii) *$\varphi = \varphi_1 - \varphi_2$, where φ_1, φ_2 are subharmonic and negative on B^n.*

PROOF. (i) $\Rightarrow$ (iii). If $0 \leq \varphi \leq h$ where h is harmonic on $B^n \setminus P$, it follows that $\varphi - h$ and $-h$ have unique subharmonic extensions to B^n, since P has vanishing Newton capacity (cf. Helms [**6**, p. 130]). Thus $\varphi = \varphi - h - (-h)$ on $B^n \setminus P$.

(iii) $\Rightarrow$ (ii). If φ is subharmonic and negative on B^n, then

$$\varphi(x) = -\int_{B^n} G(X, Y)\Delta\varphi(X) + h(X)$$

by the Riesz decomposition theorem (cf. Helms [**6**, p. 116]). It follows that if φ is the difference of two negative and subharmonic functions then (ii) holds.

(ii) $\Rightarrow$ (i). If $\psi(X) = \int_{B^n \setminus P} G(X, Y)\Delta\varphi(Y) \not\equiv +\infty$, then ψ is superharmonic and nonnegative on B^n. Hence $\varphi + \psi$ is a majorant to φ on $B^n \setminus P$, and since $\Delta\psi = -\Delta\varphi$, we have proved that (ii) implies (i), which completes the proof of the proposition.

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REMARK 1. If $f = 1/z_1$, $P = \{z \in \mathbf{C}^n, z_1 = 0\}$, then $\Delta |f|^q = q^2 |z_1|^{-(2+q)}$ so $\Delta |f|^q$ has infinite mass near P which shows that (i) in the proposition cannot hold for $|f|^q$ (cf. Cima and Graham [**4**, §4]).

3. The exceptional sets. Let E be a Borel set in $\mathbf{C}^n$. We define a set function γ_n inductively (cf. Cegrell [**2**, p.11]). If $n = 1$ then $\gamma_1 = c$ where c is the logarithmic capacity in $\mathbf{C}$. If γ_{n-1} is defined on $\mathbf{C}^{n-1}$ we define γ_n by

$$\gamma_n(E) = c(\{z_1 \in \mathbf{C}; \gamma_{n-1}(\{(z_2,\ldots,z_n); (z_1,\ldots,z_n) \in E\}) > 0\}).$$

Finally, we define $g_n(E) = \max_\tau \gamma_n(\tau E)$ where τ varies over the permutations of the coordinates. Our exceptional sets will be the sets E with $g_n(E) = 0$. Examples of such sets are pluripolar sets or, more generally, sets of vanishing Γ-capacity (cf. Ronkin [**7**] and Cegrell [**3**, p. 334]). Observe that if $g_n(E) = 0$, then E is of Lebesgue measure zero.

LEMMA. *Assume that E is a nonempty set with $g_n(E) = 0$. Then there is a point $z^0 = (z_1^0,\ldots,z_n^0) \in E$ and an m, $1 \leqslant m \leqslant n$, such that*

$$(*) \qquad c\left(\left\{z \in \mathbf{C}; \left(z_1^0,\ldots,z_{m-1}^0, z, z_{m+1}^0,\ldots,z_n\right) \in E\right\}\right) = 0.$$

PROOF. *Induction*. The case $n = 1$ is clear so assume that $(*)$ holds in dimension $\leqslant n$. If $E \subset \mathbf{C}^{n+1}$, $g_{n+1}(E) = 0$, and if no point of E satisfies $(*)$, then we want to prove that $E = \varnothing$. Now, $c(Q) = 0$, where

$$Q = \{z_1 \in \mathbf{C}; \gamma_n(\{(z_2,\ldots,z_{n+1}); (z_1,\ldots,z_{n+1}) \in E\}) > 0\}.$$

So if $z_1 \notin Q$ our induction assumption means that $(z_1, z_2,\ldots,z_{n+1}) \notin E$ for any point $(z_2,\ldots,z_{n+1}) \in \mathbf{C}^n$. Hence $E \subset Q \times \mathbf{C}^n$ so every point in E satisfies $(*)$, a contradiction, and the lemma is proved.

4. Statement and proof of the theorem.

THEOREM. *Assume that P is closed in B^n with $g_n(P) = 0$. If f is analytic on $B^n \setminus P$ and if $\Delta |f|^q$ has locally bounded mass near P, then f extends to an analytic function on B^n.*

PROOF. Without loss of generality, we can assume that $\int_{B^n \setminus P} \Delta |f|^q < +\infty$. If $n = 1$, then P is of vanishing Newton capacity. By Proposition (iii) $|f|^q = \varphi_1 - \varphi_2$, where φ_1 and φ_2 are subharmonic on B^1. It follows that $f \in L^s_{\text{loc}}(B^1)\ \forall s > 0$, then by Carleson [**1**, p. 73] or Harvey and Polking [**5**, Theorem 1.1], f extends to an analytic function.

Assume now that the theorem is proved for dimension $\leqslant n$ and let P be closed in B^{n+1} with $g_{n+1}(P) = 0$. Let $0 \leqslant \chi_\nu \in C_0^\infty(B^{n+1} \setminus P)$ be an increasing sequence of functions tending to one.

Put

$$\tau_\nu(z_m) = \int_{\mathbf{C}^n} |f|^q \sum_{s \neq m} \frac{\partial^2 \chi_\nu}{\partial z_s \partial \bar{z}_s} (dz_m)^\wedge,$$

where $(dz_m)\hat{}$ is the Lebesgue measure in $\mathbf{C}^n$. Since $|f|^q$ is plurisubharmonic on $B^{n+1}\setminus P$, τ_ν is an increasing sequence and

$$\int_{\mathbf{C}} \tau_\nu(z_m)\, dz_m \leq \int_{B^{n+1}\setminus P} \Delta |f|^q < +\infty.$$

Hence

$$\lim_{\nu \to +\infty} \tau_\nu(z_m) < +\infty, \quad \text{a.e. } (dz_m),$$

so by the induction assumption,

$$(**) \qquad (z_1, \ldots, z_{m-1}, z_{m+1}, \ldots, z_{n+1}) \mapsto f(z_1, \ldots, z_{n+1})$$

extends to an analytic function for every z_m outside a set of planar Lebesgue measure zero.

Put

$$\varphi(z) = \overline{\lim_{\substack{z' \to z \\ z' \in B^n \setminus P}}} |f(z')|, \qquad z \in B^{n+1},$$

and

$$W = \{z \in B^{n+1}; \varphi(z) < +\infty\}.$$

Then W is open, contains $B^{n+1}\setminus P$ and f has an analytic extension to W, since the function f defines a locally integrable distribution on W, and if $\chi \in C_0^\infty(W)$, $i \neq j$, then

$$\int_{\mathbf{C}^{n+1}} f \frac{\partial \chi}{\partial \bar{z}_i} = \int_{\mathbf{C}} dz_j \int_{\mathbf{C}^n} f \frac{\partial \chi}{\partial \bar{z}_i} (dz_j)\hat{} = 0$$

by $(**)$.

It remains to prove that $W = B^{n+1}$. If not, then $P' = B^{n+1}\setminus W$ is closed in B^{n+1}, nonempty and contained in P. Hence, by the lemma, there is a point $z^0 \in P'$ with property $(*)$. Since P' is closed and since $c(Q) = 0$, where

$$Q = \{z \in \mathbf{C}; (z_1^0, \ldots, z_{m-1}^0, z, z_{m+1}^0, \ldots, z_{n+1}^0) \in P'\},$$

we can choose $r > 0$ so that $T \subset\subset B^{n+1}\setminus P' = W$, where

$$T = \{(z_1^0, \ldots, z_{m-1}^0, z, z_{m+1}^0, \ldots, z_{n+1}^0);\ |z - z_m^0| = r\}.$$

Now let $z^\nu \in W$ be a sequence of points tending to z^0 so that $|f(z^\nu)| \to +\infty$, $\nu \to +\infty$. Since $|f|$ is continuous on W we can, by $(**)$, assume that $z \mapsto f(z_1^\nu, \ldots, z_{m-1}^\nu, z, z_{m+1}^\nu, \ldots, z_{n+1}^\nu)$ is analytic near $|z - z_m^\nu| \leq r$, and the maximum principle gives

$$|f(z^\nu)| \leq \sup_{|z - z_m^\nu| = r} |f(z_1^\nu, \ldots, z_{m-1}^\nu, z, z_{m+1}^\nu, \ldots, z_{n+1}^\nu)|.$$

Hence, $\overline{\lim}_{\nu \to +\infty} |f(z^\nu)| \leq \sup_{\eta \in T} |f(\eta)| < +\infty$, which is a contradiction, so P' is empty, which proves the theorem.

5. Concluding remarks.

REMARK 2. Since every analytic set is of vanishing Newton capacity and satisfies $g_n = 0$, the theorem together with the proposition gives a generalization of Theorems A and B in Cima and Graham [**4**].

REMARK 3. There are many variations of the theorem: Assume that P is closed in B^n with $g_n(P) = 0$. If f is analytic on $B^n \setminus P$ and if there is a $q > 0$ and an n-subharmonic function ψ on $B^n \setminus P$ such that $|f|^q \leqslant -\psi$ on $B^n \setminus P$, then f extends to an analytic function on B^n.

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