ON GENERALIZED FUGLEDE-PUTNAM THEOREMS OF HILBERT-SCHMIDT TYPE

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Abstract. We prove the following statements about the bounded linear operators on a separable, complex Hilbert space: (1) If A and B* are subnormal operators, and X is an invertible operator such that AX - XB \in C_2, then there exists a unitary operator U such that AU - UB \in C_2. Moreover, A*A - AA* and B*B - BB* are in C_1. (2) If A is a subnormal operator with A*A - AA* \in C_1, then for any operator X, AX - XA \in C_2 implies A*X - XA* \in C_2. (3) If A is a hyponormal contraction with 1 - AA* \in C_1, then for any operator X, AX - XA \in C_2 implies A*X - XA* \in C_2. (4) For every operator T for which T^2 is normal and T*T - TT* \in C_1, TX - XT \in C_2 implies T*X - XT* \in C_2 for any operator X. Applications of a recent result of Moore, Rogers and Trent [8] are also given.

Let H denote a separable, complex Hilbert space, and let B(H) denote the algebra of all bounded linear operators acting on H. Let K(H), C_p (0 < p < \infty) denote, respectively, the ideals of compact operators and the Schatten p-class with \| \cdot \|_p (1 \leq p < \infty) denoting the associated p-norm. Hence, C_2 is the Hilbert-Schmidt class and C_1 is the trace class. The Fuglede-Putnam theorem states that if N and M are normal operators in B(H) and NX = XM for some X \in B(H), then N*X = XM*. This theorem has been generalized [13, 7] as follows.

Theorem A. Let A, B, and X be operators on H, where A and B* are subnormal. Then AX = XB implies A*X = XB*.

In a series of papers [12-14], G. Weiss considered the Fuglede-Putnam theorem modulo certain operator ideals, and his work culminates in the following remarkable result.

Theorem B. If N, M are normal operators and X is a bounded operator, then \| NX - XM \|_2 = \| N*X - XM* \|_2. In particular, NX - XM \in C_2 implies N*X - XM* \in C_2.

The purpose of this note is to generalize Weiss' theorem to nonnormal cases. The following generalization of Theorem A was independently obtained by T. Furuta [4] after the submission of an early version of this paper.
Theorem 1. Let $A$, $B$, and $X$ be operators on $H$, where $A$ and $B^*$ are subnormal. Then $\|A^*X - XB^*\|_2 \leq \|AX - XB\|_2$. In particular, $AX - XB \in C_2$ implies $A^*X - XB^* \in C_2$.

Proof. By assumption there exists a Hilbert space $H_1$ and there exist normal operators $N$ and $M$ on $H \oplus H_1$ such that

$$N = \begin{bmatrix} A & R \\ 0 & A_1 \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} B & 0 \\ S & B_1 \end{bmatrix}. $$

Let $Y = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. Now

$$NY - YM = \begin{bmatrix} AX - XB & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad N^*Y - YM^* = \begin{bmatrix} A^*X - XB^* & -XS^* \\ R^*X & 0 \end{bmatrix}. $$

By Theorem B we have $\|NY - YM\|_2 = \|N^*Y - YM^*\|_2$. Therefore

$$\|AX - XB\|_2^2 = \|A^*X - XB^*\|_2^2 + \|XS^*\|_2^2 + \|R^*X\|_2^2.$$ 

Hence, $\|A^*X - XB^*\|_2 \leq \|AX - XB\|_2$.

Corollary 1. If $A$, $B$, and $X$ are operators on $H$ such that $A$ and $B^*$ are subnormal and $X$ is invertible, and if $AX - XB \in C_2$, then there exists a unitary operator $U$ such that $AU - UB \in C_2$. Moreover, $AA^* - A^*A$ and $BB^* - B^*B$ are in $C_1$.

Proof. By the proof of Theorem 1, we see that $A^*X - XB^*$, $XS^*$ and $R^*X$ are in $C_2$. Since $N$ and $M$ are normal operators and $X$ is invertible, we get $AA^* - A^*A = RR^* \in C_1$ and $BB^* - B^*B = S^*S \in C_1$. But $AX - XB \in C_2$ and $A^*X - XB^* \in C_2$ imply that $X^*XB - BX^*X \in C_2$. Let $X = UP$ be the polar decomposition of $X$. Thus $P^2B - BP^2 \in C_2$. Since $P \geq 0$ and invertible, it follows (see [12]) that $PB - BP \in C_2$. Therefore, $AUP - UBP \in C_2$ implies that $AUP - UBP \in C_2$. But the invertibility of $P$ implies that $AU - UB \in C_2$, as required.

The following corollary can be proved by an argument similar to the one above.

Corollary 2. If $A$, $B$, and $X$ are operators on $H$ such that $A$ and $B^*$ are subnormal and $X$ is invertible and positive, and if $AX - XB \in C_2$, then $A - B \in C_2$ and $AA^* - A^*A$ and $BB^* - B^*B$ are in $C_1$.

Kulkarni [7] and Stampfli [11] each gave an example of operators $A$, $B$, and $X$ on a Hilbert space such that $A$ and $B$ are subnormal with $AX = XB$, but $A^*X \neq XB^*$.

We now give an example of subnormal operators $A$ and $B$ such that $AX - XB \in C_2$, but $A^*X - XB^* \not\in C_2$.

Example 1. Let $H = \bigoplus_{n=1}^{\infty} H_n$, where $H_n = H$ for all $n$. Let $A$ be the operator valued weighted shift

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \vdots & \ddots \end{bmatrix}. $$
Let
\[
X = \begin{bmatrix}
0 & 0 & 0 & 0 \\
P & 0 & 0 & 0 \\
0 & P & 0 & 0 \\
0 & 0 & P & 0 \\
\end{bmatrix},
\]
where \(P\) is a projection with \(P \notin C_2\), and let
\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
P & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{bmatrix}.
\]

It is clear that \(A\) is subnormal and \(AX - XB = 0 \in C_2\), but \(A^*X - XB^* = \text{diag}(P, 0, 0, \ldots) \notin C_2\). The fact that \(B\) is subnormal follows from Theorem 3.6 in [6].

If, however, \(A\) is subnormal with \(A^*A - AA^* \in C_1\), then we have

**Theorem 2.** If \(A \in B(H)\) is subnormal with \(A^*A - AA^* \in C_1\), then for \(X \in B(H)\),
\(AX -XA \in C_2\) implies that \(A^*X -XA^* \in C_2\).

**Proof.** Let
\[
N = \begin{bmatrix}
A & R \\
0 & A_1 \\
\end{bmatrix}
\]
be a normal extension of \(A\). Let \(Y = \begin{bmatrix} X & 0 \end{bmatrix}\). Then
\[
NY -YN = \begin{bmatrix} AX -XA & -XR \\
0 & 0 \\
\end{bmatrix} \quad \text{and} \quad N^*Y -YN^* = \begin{bmatrix} A^*X -XA^* & 0 \\
R^*X & 0 \\
\end{bmatrix}.
\]
Since \(N\) is normal, it follows that \(A^*A - AA^* = RR^* \in C_1\). Therefore \(R \in C_2\) and so \(NY -YN \in C_2\). Hence, by Theorem B, \(N^*Y -YN^* \in C_2\), from which it follows that \(A^*X -XA^* \in C_2\).

**Corollary 3.** If \(A \in B(H)\) is a subnormal and rationally cyclic operator, then for \(X \in B(H)\), \(AX -XA \in C_2\) implies \(A^*X -XA^* \in C_2\).

**Proof.** A theorem of Berger and Shaw [2] states that if \(A\) is a rationally cyclic hyponormal operator, then \(A^*A - AA^* \in C_1\).

**Lemma 1.** If \(A \in B(H)\) is a hyponormal contraction with \(1 - AA^* \in C_1\), then \(1 - A^*A \in C_1\).

**Proof.** For any orthonormal basis \(\{e_n\}\) of \(H\) we have
\[
\sum_{n=1}^{\infty} ((1 - A^*A)e_n, e_n) = \sum_{n=1}^{\infty} 1 - (A^*Ae_n, e_n) \\
\leq \sum_{n=1}^{\infty} 1 - (AA^*e_n, e_n) = \sum_{n=1}^{\infty} ((1 - AA^*)e_n, e_n) < \infty.
\]
Since \(1 - A^*A \geq 0\), it follows that \(1 - A^*A \in C_1\).
**Theorem 3.** If \( A \in B(H) \) is a hyponormal contraction with \( 1 - AA^* \in C_1 \), then for \( X \in B(H) \), \( AX -XA \in C_2 \) implies \( A^*X -XA^* \in C_2 \).

**Proof.** Let

\[
N = \begin{bmatrix} A & (1 - AA^*)^{1/2} \\ (1 - A^*A)^{1/2} & -A^* \end{bmatrix}
\text{ on } H \oplus H.
\]

Then it is easy to see that \( N \) is unitary [5]. Let \( Y = \begin{bmatrix} X_0 & 0 \\ 0 & 0 \end{bmatrix} \). Then

\[
NY - YN = \begin{bmatrix} AX -XA & -X(1 - A^*A)^{1/2} \\ (1 - AA^*)^{1/2} S & 0 \end{bmatrix}.
\]

By Lemma 1, we have \( 1 - A^*A \) and \( 1 - AA^* \in C_1 \). Therefore \( (1 - A^*A)^{1/2} \) and \( (1 - AA^*)^{1/2} \in C_2 \) and so \( NY - YN \in C_2 \). Hence, by Theorem B,

\[
N^*Y - YN^* = \begin{bmatrix} A^*X -XA^* & -X(1 - A^*A)^{1/2} \\ (1 - AA^*)^{1/2} X & 0 \end{bmatrix} \in C_2.
\]

Hence, \( A^*X -XA^* \in C_2 \).

**Corollary 4.** If \( V \) is an isometry of finite multiplicity in \( B(H) \), then for \( X \in B(H) \), \( VX - XV \in C_2 \) implies \( V^*X - XV^* \in C_2 \). In particular this is the case when \( V \) is a unilateral shift of finite multiplicity.

**Theorem 4.** If \( T \in B(H) \) with \( T^2 \) normal and \( T^*T - TT^* \in C_1 \), then for \( X \in B(H) \), \( TX -XT \in C_2 \) implies \( T^*X - XT^* \in C_2 \).

**Proof.** By Radjavi's and Rosenthal's model [10],

\[
T = \begin{bmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{bmatrix},
\]

where \( A, B \) are normal operators, \( C \not\in 0 \) and one-to-one, \( BC = CB \), and \( \sigma(B) \subset \text{closed upper half-plane} \). But \( T^*T - TT^* \in C_1 \) implies (easy matrix computations) that \( C^2 \in C_1 \). Hence, \( C \in C_2 \). Therefore \( T = N + K \), where

\[
N = \begin{bmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -B \end{bmatrix}
\]

is normal and

\[
K = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{bmatrix} \in C_2.
\]

Now \( TX -XT \in C_2 \) implies that \( NX - XN \in C_2 \) and so, by Theorem B, \( N^*X - XN^* \in C_2 \). Therefore

\[
T^*X - XT^* = N^*X - XN^* + K^*X - XK^* \in C_2.
\]

We now give an example to show that \( T^*T - TT^* \in C_1 \) is necessary for Theorem 4 to hold.
Example 2. If $H_0 = H \oplus H$, $T = \begin{bmatrix} 0 & D \\ D & 0 \end{bmatrix}$ on $H_0$, and $X = \begin{bmatrix} A \\ D \end{bmatrix}$, where $A$ is arbitrary in $B(H)$ and $D = \text{diag}(1, 1/\sqrt{2}, 1/\sqrt{3}, \ldots)$, then $T^2 = 0$, $TX - XT = 0$, but

$$T^*X - XT^* = \begin{bmatrix} D & 0 \\ 0 & D \end{bmatrix} \notin C_2.$$

Remark. It is not known whether these results are valid for the case $1 \leq p \neq 2$. Theorem B is not valid for $C_p$, $0 < p < 1$, and $F(H)$ (the ideal of finite rank operators) [14]. It is easy to see, however, that they are valid with $K(H)$ in place of $C_2$. (See [14] for Theorem B in this case.)

The most recent generalization of the Fuglede-Putnam theorem was obtained by Moore, Rogers and Trent [8] and can be stated as follows.

**Theorem 5.** If $A$ and $B^*$ are hyponormal operators in $B(H)$, then for $X \in B(H)$, $AX = XB$ implies $A^*X = XB^*$.

An attempt to generalize Theorem B to the hyponormal case was made by T. Furuta [4], who obtained the following result.

**Theorem 6.** If $A$ and $B^*$ are hyponormal operators in $B(H)$, then for any $X \in C_2$, $\|A^*X - XB^*\|_2 \leq \|AX - XB\|_2$.

Whether Theorem 6 can be relaxed is not yet known. As an application of the above results we prove

**Lemma 2.** Let $V$, $A$ and $X$ be operators in $B(H)$. If $V$ is an isometry, $A^*$ is hyponormal, and $X$ is one-to-one, then $VX =XA$ implies $A$ is unitary.

**Proof.** By Theorem 5, $VX = AX$ implies that $V^*X = XA^*$. Multiply the first equation on the left by $V^*$ to get $X = V^*XA$. Therefore, $X = XA^*A$. The fact that $X$ is one-to-one implies that $1 = A^*A$. Since $A^*$ is hyponormal and $A^*A = 1$, it follows that $A$ is normal and, hence, unitary.

We use a result of Brown, Douglas and Fillmore [1] to obtain

**Theorem 7.** Let $V$ and $A$ be as in Lemma 2, and let $X$ be invertible. If $VX -XA \in K(H)$, then $A$ is either a compact perturbation of a unitary operator or the adjoint of a shift of finite multiplicity $n$.

**Proof.** Let $B(H)/K(H)$ be the Calkin algebra and $\pi: B(H) \to B(H)/K(H)$ the quotient map. Since $\pi(V)\pi(X) = \pi(X)\pi(A)$, it follows by the remark following Example 2 that $\pi(V^*)\pi(X) = \pi(X)\pi(X)^*$. Using the same argument that was used in Lemma 2, we show that $\pi(A)$ is unitary. Since $A^*$ is hyponormal (has a nonnegative Fredholm index), the result now follows by using Theorem 3.1 of [1].

Utilizing Theorem 5 we can also generalize and give, to some extent, a different proof of Theorem 3.9.1 in [9] as follows.

**Theorem 8.** Let $T = A + iB$ be the Cartesian decomposition of a hyponormal operator $T \in B(H)$. If $AB$ is hyponormal, then $T$ is normal.
Proof. Let $Q = AB$. Then $QA = AQ^*$. Since $T$ is hyponormal, $i(Q - Q^*) = i(AB - BA) \geq 0$. Now applying Theorem 5 gives $Q^*A = AQ$. Thus, $(Q + Q^*)A = A(Q + Q^*)$ and $(Q - Q^*)A = A(Q^* - Q)$. Let $Y = i(Q - Q^*)$. Then $YA = -AY$ and so $Y^2A = AY^2$. Since $Y \geq 0$, the spectral theorem now implies that $YA = AY$. Therefore, $QA = AQ$ and so $A(AB - BA) = (AB - BA)A = 0$. Hence, $\sigma(AB - BA) = 0 [9]$. Since $AB - BA$ is normal (skew hermitian), it follows that $AB - BA = 0$ and, hence, $T$ is normal.

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References


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