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ON GENERALIZED FUGLEDE-PUTNAM THEOREMS OF HILBERT-SCHMIDT TYPE

FUAD KITTANEH

ABSTRACT. We prove the following statements about the bounded linear operators on a separable, complex Hilbert space: (1) If A and B^* are subnormal operators, and X is an invertible operator such that $AX - XB \in C_2$, then there exists a unitary operator U such that $AU - UB \in C_2$. Moreover, $A^*A - AA^*$ and $B^*B - BB^*$ are in C_1. (2) If A is a subnormal operator with $A^*A - AA^* \in C_1$, then for any operator X, $AX - XA \in C_2$ implies $A^*X - XA^* \in C_2$. (3) If A is a hyponormal contraction with $1 - AA^* \in C_1$, then for any operator X, $AX - XA \in C_2$ implies $A^*X - XA^* \in C_2$. (4) For every operator T for which T^2 is normal and $T^*T - TT^* \in C_1$, $TX - XT \in C_2$ implies $T^*X - XT^* \in C_2$ for any operator X. Applications of a recent result of Moore, Rogers and Trent [**8**] are also given.

Let H denote a separable, complex Hilbert space, and let $B(H)$ denote the algebra of all bounded linear operators acting on H. Let $K(H)$, C_p $(0 < p < \infty)$ denote, respectively, the ideals of compact operators and the Schatten p-class with $\|\cdot\|_p$ $(1 \leqslant p < \infty)$ denoting the associated p-norm. Hence, C_2 is the Hilbert-Schmidt class and C_1 is the trace class. The Fuglede-Putnam theorem states that if N and M are normal operators in $B(H)$ and $NX = XM$ for some $X \in B(H)$, then $N^*X = XM^*$. This theorem has been generalized [**13, 7**] as follows.

THEOREM A. *Let A, B, and X be operators on H, where A and B^* are subnormal. Then $AX = XB$ implies $A^*X = XB^*$.*

In a series of papers [**12–14**], G. Weiss considered the Fuglede-Putnam theorem modulo certain operator ideals, and his work culminates in the following remarkable result.

THEOREM B. *If N, M are normal operators and X is a bounded operator, then $\|NX - XM\|_2 = \|N^*X - XM^*\|_2$. In particular, $NX - XM \in C_2$ implies $N^*X - XM^* \in C_2$.*

The purpose of this note is to generalize Weiss' theorem to nonnormal cases. The following generalization of Theorem A was independently obtained by T. Furuta [**4**] after the submission of an early version of this paper.

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THEOREM 1. *Let A, B, and X be operators on H, where A and B^* are subnormal. Then $\|A^*X - XB^*\|_2 \leqslant \|AX - XB\|_2$. In particular, $AX - XB \in C_2$ implies $A^*X - XB^* \in C_2$.*

PROOF. By assumption there exists a Hilbert space H_1 and there exist normal operators N and M on $H \oplus H_1$ such that

$$N = \begin{vmatrix} A & R \\ 0 & A_1 \end{vmatrix} \quad \text{and} \quad M = \begin{vmatrix} B & 0 \\ S & B_1 \end{vmatrix}.$$

Let $Y = \begin{vmatrix} X & 0 \\ 0 & 0 \end{vmatrix}$. Now

$$NY - YM = \begin{vmatrix} AX - XB & 0 \\ 0 & 0 \end{vmatrix} \quad \text{and} \quad N^*Y - YM^* = \begin{vmatrix} A^*X - XB^* & -XS^* \\ R^*X & 0 \end{vmatrix}.$$

By Theorem B we have $\|NY - YM\|_2 = \|N^*Y - YM^*\|_2$. Therefore

$$\|AX - XB\|_2^2 = \|A^*X - XB^*\|_2^2 + \|XS^*\|_2^2 + \|R^*X\|_2^2.$$

Hence, $\|A^*X - XB^*\|_2 \leqslant \|AX - XB\|_2$.

COROLLARY 1. *If A, B, and X are operators on H such that A and B^* are subnormal and X is invertible, and if $AX - XB \in C_2$, then there exists a unitary operator U such that $AU - UB \in C_2$. Moreover, $AA^* - A^*A$ and $BB^* - B^*B$ are in C_1.*

PROOF. By the proof of Theorem 1, we see that $A^*X - XB^*$, XS^* and R^*X are in C_2. Since N and M are normal operators and X is invertible, we get $AA^* - A^*A = RR^* \in C_1$ and $BB^* - B^*B = S^*S \in C_1$. But $AX - XB \in C_2$ and $A^*X - XB^* \in C_2$ imply that $X^*XB - BX^*X \in C_2$. Let $X = UP$ be the polar decomposition of X. Thus $P^2B - BP^2 \in C_2$. Since $P \geqslant 0$ and invertible, it follows (see [**12**]) that $PB - BP \in C_2$. Therefore, $AUP - UPB \in C_2$ implies that $AUP - UBP \in C_2$. But the invertibility of P implies that $AU - UB \in C_2$, as required.

The following corollary can be proved by an argument similar to the one above.

COROLLARY 2. *If A, B, and X are operators on H such that A and B^* are subnormal and X is invertible and positive, and if $AX - XB \in C_2$, then $A - B \in C_2$ and $AA^* - A^*A$ and $BB^* - B^*B$ are in C_1.*

Kulkarni [**7**] and Stampfli [**11**] each gave an example of operators A, B, and X on a Hilbert space such that A and B are subnormal with $AX = XB$, but $A^*X \neq XB^*$.

We now give an example of subnormal operators A and B such that $AX - XB \in C_2$, but $A^*X - XB^* \notin C_2$.

EXAMPLE 1. Let $H = \oplus_{n=1}^{\infty} H_n$, where $H_n = H$ for all n. Let A be the operator valued weighted shift

$$A = \begin{vmatrix} 0 & 0 & 0 & 0 & \\ 1 & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & \ddots & \ddots \end{vmatrix}.$$

Let

$$X = \begin{vmatrix} 0 & 0 & 0 & 0 & \\ P & 0 & 0 & 0 & \\ 0 & P & 0 & 0 & \\ 0 & 0 & P & 0 & \\ & & & \ddots & \ddots \end{vmatrix},$$

where P is a projection with $P \notin C_2$, and let

$$B = \begin{vmatrix} 0 & 0 & 0 & 0 & \\ P & 0 & 0 & 0 & \\ 0 & 1 & 0 & 0 & \\ 0 & 0 & 1 & 0 & \\ & & & \ddots & \ddots \end{vmatrix}.$$

It is clear that A is subnormal and $AX - XB = 0 \in C_2$, but $A^*X - XB^* = \operatorname{diag}(P, 0, 0, \dots) \notin C_2$. The fact that B is subnormal follows from Theorem 3.6 in [**6**].

If, however, A is subnormal with $A^*A - AA^* \in C_1$, then we have

THEOREM 2. *If $A \in B(H)$ is subnormal with $A^*A - AA^* \in C_1$, then for $X \in B(H)$, $AX - XA \in C_2$ implies that $A^*X - XA^* \in C_2$.*

PROOF. Let

$$N = \begin{vmatrix} A & R \\ 0 & A_1 \end{vmatrix}$$

be a normal extension of A. Let $Y = \left|\begin{smallmatrix} X & 0 \\ 0 & 0 \end{smallmatrix}\right|$. Then

$$NY - YN = \begin{vmatrix} AX - XA & -XR \\ 0 & 0 \end{vmatrix} \quad \text{and} \quad N^*Y - YN^* = \begin{vmatrix} A^*X - XA^* & 0 \\ R^*X & 0 \end{vmatrix}.$$

Since N is normal, it follows that $A^*A - AA^* = RR^* \in C_1$. Therefore $R \in C_2$ and so $NY - YN \in C_2$. Hence, by Theorem B, $N^*Y - YN^* \in C_2$, from which it follows that $A^*X - XA^* \in C_2$.

COROLLARY 3. *If $A \in B(H)$ is a subnormal and rationally cyclic operator, then for $X \in B(H)$, $AX - XA \in C_2$ implies $A^*X - XA^* \in C_2$.*

PROOF. A theorem of Berger and Shaw [**2**] states that if A is a rationally cyclic hyponormal operator, then $A^*A - AA^* \in C_1$.

LEMMA 1. *If $A \in B(H)$ is a hyponormal contraction with $1 - AA^* \in C_1$, then $1 - A^*A \in C_1$.*

PROOF. For any orthonormal basis $\{e_n\}$ of H we have

$$\sum_{n=1}^{\infty} ((1 - A^*A)e_n, e_n) = \sum_{n=1}^{\infty} 1 - (A^*Ae_n, e_n)$$
$$\leqslant \sum_{n=1}^{\infty} 1 - (AA^*e_n, e_n) = \sum_{n=1}^{\infty} ((1 - AA^*)e_n, e_n) < \infty.$$

Since $1 - A^*A \geqslant 0$, it follows that $1 - A^*A \in C_1$.

THEOREM 3. *If $A \in B(H)$ is a hyponormal contraction with $1 - AA^* \in C_1$, then for $X \in B(H)$, $AX - XA \in C_2$ implies $A^*X - XA^* \in C_2$.*

PROOF. Let

$$N = \begin{vmatrix} A & (1 - AA^*)^{1/2} \\ (1 - A^*A)^{1/2} & -A^* \end{vmatrix} \quad \text{on } H \oplus H.$$

Then it is easy to see that N is unitary [**5**]. Let $Y = \begin{vmatrix} X & 0 \\ 0 & 0 \end{vmatrix}$. Then

$$NY - YN = \begin{vmatrix} AX - XA & -X(1 - A^*A)^{1/2} \\ (1 - AA^*)^{1/2}S & 0 \end{vmatrix}.$$

By Lemma 1, we have $1 - A^*A$ and $1 - AA^* \in C_1$. Therefore $(1 - A^*A)^{1/2}$ and $(1 - AA^*)^{1/2} \in C_2$ and so $NY - YN \in C_2$. Hence, by Theorem B,

$$N^*Y - YN^* = \begin{vmatrix} A^*X - XA^* & -X(1 - A^*A)^{1/2} \\ (1 - AA^*)^{1/2}X & 0 \end{vmatrix} \in C_2.$$

Hence, $A^*X - XA^* \in C_2$.

COROLLARY 4. *If V is an isometry of finite multiplicity in $B(H)$, then for $X \in B(H)$, $VX - XV \in C_2$ implies $V^*X - XV^* \in C_2$. In particular this is the case when V is a unilateral shift of finite multiplicity.*

THEOREM 4. *If $T \in B(H)$ with T^2 normal and $T^*T - TT^* \in C_1$, then for $X \in B(H)$, $TX - XT \in C_2$ implies $T^*X - XT^* \in C_2$.*

PROOF. By Radjavi's and Rosenthal's model [**10**],

$$T = \begin{vmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & 0 & -B \end{vmatrix},$$

where A, B are normal operators, $C \geqslant 0$ and one-to-one, $BC = CB$, and $\sigma(B) \subset$ closed upper half-plane. But $T^*T - TT^* \in C_1$ implies (easy matrix computations) that $C^2 \in C_1$. Hence, $C \in C_2$. Therefore $T = N + K$, where

$$N = \begin{vmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & -B \end{vmatrix}$$

is normal and

$$K = \begin{vmatrix} 0 & 0 & 0 \\ 0 & 0 & C \\ 0 & 0 & 0 \end{vmatrix} \in C_2.$$

Now $TX - XT \in C_2$ implies that $NX - XN \in C_2$ and so, by Theorem B, $N^*X - XN^* \in C_2$. Therefore

$$T^*X - XT^* = N^*X - XN^* + K^*X - XK^* \in C_2.$$

We now give an example to show that $T^*T - TT^* \in C_1$ is necessary for Theorem 4 to hold.

EXAMPLE 2. If $H_0 = H \oplus H$, $T = \begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}$ on H_0, and $X = \begin{vmatrix} A & D \\ 0 & A \end{vmatrix}$, where A is arbitrary in $B(H)$ and $D = \operatorname{diag}(1, 1/\sqrt{2}, 1/\sqrt{3}, \ldots)$, then $T^2 = 0$, $TX - XT = 0$, but

$$T^*X - XT^* = \begin{vmatrix} -D & 0 \\ 0 & D \end{vmatrix} \notin C_2.$$

REMARK. It is not known whether these results are valid for the case $1 \leqslant p \neq 2$. Theorem B is not valid for C_p, $0 < p < 1$, and $F(H)$ (the ideal of finite rank operators) [**14**]. It is easy to see, however, that they are valid with $K(H)$ in place of C_2. (See [**14**] for Theorem B in this case.)

The most recent generalization of the Fuglede-Putnam theorem was obtained by Moore, Rogers and Trent [**8**] and can be stated as follows.

THEOREM 5. *If A and B^* are hyponormal operators in $B(H)$, then for $X \in B(H)$, $AX = XB$ implies $A^*X = XB^*$.*

An attempt to generalize Theorem B to the hyponormal case was made by T. Furuta [**4**], who obtained the following result.

THEOREM 6. *If A and B^* are hyponormal operators in $B(H)$, then for any $X \in C_2$, $\|A^*X - XB^*\|_2 \leqslant \|AX - XB\|_2$.*

Whether Theorem 6 can be relaxed is not yet known. As an application of the above results we prove

LEMMA 2. *Let V, A and X be operators in $B(H)$. If V is an isometry, A^* is hyponormal, and X is one-to-one, then $VX = XA$ implies A is unitary.*

PROOF. By Theorem 5, $VX = XA$ implies that $V^*X = XA^*$. Multiply the first equation on the left by V^* to get $X = V^*XA$. Therefore, $X = XA^*A$. The fact that X is one-to-one implies that $1 = A^*A$. Since A^* is hyponormal and $A^*A = 1$, it follows that A is normal and, hence, unitary.

We use a result of Brown, Douglas and Fillmore [**1**] to obtain

THEOREM 7. *Let V and A be as in Lemma 2, and let X be invertible. If $VX - XA \in K(H)$, then A is either a compact perturbation of a unitary operator or the adjoint of a shift of finite multiplicity n.*

PROOF. Let $B(H)/K(H)$ be the Calkin algebra and π: $B(H) \to B(H)/K(H)$ the quotient map. Since $\pi(V)\pi(X) = \pi(X)\pi(A)$, it follows by the remark following Example 2 that $\pi(V)^*\pi(X) = \pi(X)\pi(X)^*$. Using the same argument that was used in Lemma 2, we show that $\pi(A)$ is unitary. Since A^* is hyponormal (has a nonnegative Fredholm index), the result now follows by using Theorem 3.1 of [**1**].

Utilizing Theorem 5 we can also generalize and give, to some extent, a different proof of Theorem 3.9.1 in [**9**] as follows.

THEOREM 8. *Let $T = A + iB$ be the Cartesian decomposition of a hyponormal operator $T \in B(H)$. If AB is hyponormal, then T is normal.*

PROOF. Let $Q = AB$. Then $QA = AQ^*$. Since T is hyponormal, $i(Q - Q^*) = i(AB - BA) \geqslant 0$. Now applying Theorem 5 gives $Q^*A = AQ$. Thus, $(Q + Q^*)A = A(Q + Q^*)$ and $(Q - Q^*)A = A(Q^* - Q)$. Let $Y = i(Q - Q^*)$. Then $YA = -AY$ and so $Y^2A = AY^2$. Since $Y \geqslant 0$, the spectral theorem now implies that $YA = AY$. Therefore, $QA = AQ$ and so $A(AB - BA) = (AB - BA)A = 0$. Hence, $\sigma(AB - BA) = 0$ [**9**]. Since $AB - BA$ is normal (skew hermitian), it follows that $AB - BA = 0$ and, hence, T is normal.

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47405

Current address: Department of Mathematics, United Arab Emirates University, P. O. Box 15551 Al-Ain, U.A.E.