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ZERO-FREE PARABOLIC REGIONS FOR POLYNOMIALS WITH COMPLEX COEFFICIENTS

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ABSTRACT. Recent results by P. Henrici, E. B. Saff and R. S. Varga on zero-free parabolic regions for sequences of polynomials generated from three-term recurrence relations with real coefficients are generalized to complex coefficients by continued fraction methods. Especially, it is shown that all zeros of the generalized Bessel polynomials $Y_n^{(\delta)}$ for complex δ are contained in a cardioid region, which generalizes a result of E. B. Saff and R. S. Varga for real δ.

1. Introduction. In [**6**] E. B. Saff and R. S. Varga showed that certain sequences of polynomials defined by three-term recurrence relations with real coefficients have no zeros in a parabolic region. From this it is deduced (cf. [**2**, pp. 75–89]) that all zeros of the mth generalized Bessel polynomial $Y_m^{(\delta)}(z)$, depending on the parameter δ, are contained in the cardioid region

$$\{z = re^{i\vartheta} \in \mathbf{C}\colon 0 < r < (1+\cos\vartheta)/(m+1+\delta), |\vartheta| < \pi\} \cup \{2/(m+1+\delta)\},$$

provided $m + 1 + \delta > 0$. In [**3**] P. Henrici generalized the results of [**6**] so as to apply to interpolation polynomials with real interpolation points. In the present paper the continued fraction method of P. Henrici [**3**] is generalized so as to yield zero-free parabolic regions also for polynomials satisfying three-term recursion relations with *complex* coefficients, which include interpolation polynomials with complex interpolation points. As an interesting application, it is shown that all zeros of $Y_m^{(\delta)}(z)$, $m \geq 2$, are contained in the open cardioid region

$$\{z = re^{i\vartheta} \in \mathbf{C}\colon 0 < r < (1+\cos\vartheta)/|m+1+\delta|\cos^2(\chi/2), |\vartheta| < \pi\},$$

where $m + 1 + \delta = |m+1+\delta|e^{i\chi} \neq 0$ and $|\chi| < \pi$, i.e. δ is complex, provided $m + 1 + \delta$ is not ≤ 0. In particular, for $0 < |\chi| < \pi/2$ this region is properly contained in the open cardioid region

$$\{z = re^{i\vartheta} \in \mathbf{C}\colon 0 < r < (1+\cos\vartheta)/(m+1+\operatorname{Re}\delta), |\vartheta| < \pi\}.$$

2. The basic continued fraction method. More generally than in [**3**] we define

$$q_n := (b_n + z_{n+1})q_{n-1} - a_n z_n q_{n-2}, \qquad n \in \mathbf{N}, q_{-1} := 0, q_0 := 1, \tag{1}$$

where $a_n, b_n, z_n \in \mathbf{C}$, $n \in \mathbf{N}$ and $a_n = |a_n|e^{i\psi_n} \neq 0$, $n \geq 2$.

Since $q_{-1} = 0$, all q_n, $n \geq 1$, are independent of a_1 and z_1. We want to determine conditions on a_n, b_n, z_n which imply $q_n \neq 0$ for all $n \in \mathbf{N}$ or $1 \leq n \leq N$. Similarly as

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in [**3**] we observe that q_n is the denominator of the continued fraction

$$(2) \qquad w_n = \frac{p_n}{q_n} = \frac{1}{b_1 + z_2} - \frac{z_2 a_2}{b_2 + z_3} - \cdots - \frac{z_n a_n}{b_n + z_{n+1}}.$$

Since p_n, q_n cannot vanish simultaneously if all $a_n z_n \neq 0$, we first assume $z_n \neq 0$ for $n \geqslant 2$. Later on we will see that all results below remain valid if some $z_n = 0$. It then suffices to show that $w_n \neq \infty$ in order to obtain $q_n \neq 0$. (2) can be written as $w_n = s_1 \circ \cdots \circ s_n(z_{n+1})$, where ($\overline{\mathbf{C}}$ is the closed complex plane)

$$(3) \qquad \begin{aligned} s_n(u) &:= z_n(1 - (a_n/(b_n + u))), \qquad n \geqslant 2, \quad \text{and} \\ s_1(u) &:= 1/(b_1 + u), \qquad u \in \overline{\mathbf{C}}. \end{aligned}$$

We want to determine closed half-planes $H_n \subset \overline{\mathbf{C}}$, $n \in \mathbf{N}$, and (large) sets $P_n \subset \mathbf{C}$, $n \geqslant 2$, such that for $z_n \in P_n$,

$$(4) \qquad D_n := s_n(H_n) \subset H_{n-1} \text{ holds and } D_n \subset \mathbf{C} \text{ is a finite closed disk, } n \geqslant 2.$$

Assume, in addition, that $-b_1 \notin H_1$ (or more generally $-b_1 \notin D_2$). Then (cf. (8)) $z_n \in P_n$, $n \geqslant 2$, and (4) imply $w_n \neq \infty$ and, hence, $q_n \neq 0$ for $n \geqslant 1$. We now define, for $n \in \mathbf{N}$,

$$(5) \quad H_n := \{\zeta \in \mathbf{C}\colon \operatorname{Re} e^{i\varphi_n}(\zeta + d_n) \geqslant 0\}, \qquad \varphi_n \in \mathbf{R}, d_n \in \mathbf{C}; \operatorname{Re} e^{i\varphi_n} d_n > 0.$$

Since $s_n(-b_n) = \infty$, D_n is a finite disk iff $-b_n \notin H_n$ or $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$, which we want to assume for $n \geqslant 2$. Next, we evaluate D_n explicitly. Let b_n^* be symmetric to $-b_n$ with respect to ∂H_n, the boundary of H_n. Then $b_n^* = -d_n + e^{-2i\varphi_n}(\bar{b}_n - \bar{d}_n)$, and the center of D_n is given by $\xi_n := s_n(b_n^*) = z_n(1 - c_n)$, where

$$(6) \qquad c_n := a_n/(b_n + b_n^*) = e^{i\varphi_n} a_n / 2 \operatorname{Re} e^{i\varphi_n}(b_n - d_n), \qquad n \geqslant 2.$$

Since $z_n = s_n(\infty) \in \partial D_n$, the boundary of D_n, the radius of D_n is given by $r_n := |z_n - \xi_n| = |z_n c_n|$. Hence

$$(7) \qquad D_n = \{\zeta \in \mathbf{C}\colon |\zeta - z_n + z_n c_n| \leqslant |z_n c_n|\}, \qquad n \geqslant 2.$$

For $n \geqslant 2$, $D_n \subset H_{n-1}$ is equivalent to $\operatorname{Re} e^{i\varphi_{n-1}}(\xi_n + d_{n-1}) - r_n \geqslant 0$ or, explicitly,

$$\operatorname{Re} e^{i\varphi_{n-1}}(z_n - z_n c_n + d_{n-1}) - |z_n c_n| \geqslant 0,$$

or

$$(8) \qquad \operatorname{Re} e^{i\varphi_{n-1}}(z_n + d_{n-1}) \geqslant \operatorname{Re}(e^{i\varphi_{n-1}} z_n c_n) + |z_n c_n|, \qquad n \geqslant 2.$$

Since here the right side is $\geqslant 0$, $z_n \in H_{n-1}$ follows and, hence, $P_n \subset H_{n-1}$ must hold for $n \geqslant 2$. We now define P_n by

$$(9) \quad P_n := \{\zeta \in \mathbf{C}\colon \operatorname{Re}(e^{i\varphi_{n-1}} \zeta c_n) + |\zeta c_n| \leqslant 2h(c_n) \operatorname{Re} e^{i\varphi_{n-1}}(\zeta + d_{n-1})\},$$

where the real-valued function h satisfies $0 < h(c_n) \leqslant 1/2$, $n \geqslant 2$.

Obviously (8) holds for each $z_n \in P_n$, and different choices of the function h lead to different conditions on a_n, b_n, z_n. Next, $a_n \neq 0$ implies $c_n \neq 0$. Hence (9) is equivalent to

$$(10)\qquad P_n = \{\zeta \in \mathbf{C}: |\zeta| \leqslant \operatorname{Re}(e^{i\varphi_{n-1}}\zeta(2h(c_n) - c_n)/|c_n|) + 2(h(c_n)/|c_n|)\operatorname{Re} e^{i\varphi_{n-1}}d_{n-1}\},$$

where $P_n \subset H_{n-1}$, $n \geqslant 2$. Observe that each P_n is convex. Finally, assume that $z_1, \ldots, z_m \neq 0$, $z_{m+1} = \cdots = z_{m+j} = 0$, $z_{m+j+1} \neq 0$ for some $m, j \geqslant 1$. Then $q_1, \ldots, q_m \neq 0$ by the preceding considerations and, by (1),

$$q_{m+\nu} = (b_{m+\nu} + z_{m+\nu+1})q_{m+\nu-1}, \qquad 1 \leqslant \nu \leqslant j,$$

which is $\neq 0$ because $z_{m+\nu+1} \in P_{m+\nu+1} \subset H_{m+\nu}$ and $-b_{m+\nu} \notin H_{m+\nu}$. For $n \in \mathbf{N}$ then $q_{m+j+n-1} = q_{m+j-1}q_n^*$ holds, where q_n^* satisfies

$$q_n^* = (b_{(m+j-1)+n} + z_{(m+j-1)+n+1})q_{n-1}^* - a_{(m+j-1)+n}z_{(m+j-1)+n}q_{n-2}^*$$

with $q_{-1}^* = 0$, $q_0^* = 1$. To these q_n^* the above considerations can be applied until, again for some n, $z_{(m+j-1)+n} = 0$ holds. But then the preceding argument can be repeated.

3. General results. In the preceding considerations we have proved

THEOREM 1. *If (cf.* (1),(5),(9)) $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$, $1 \leqslant n \leqslant N$ *and* $z_n \in P_n$, $2 \leqslant n \leqslant N+1$, *or if, more generally*, $|b_1 + z_2 - z_2c_2| > |z_2c_2|$ (*i.e.* $-b_1 \notin D_2$), $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$, $2 \leqslant n \leqslant N$ *and* $z_n \in P_n$, $3 \leqslant n \leqslant N+1$, *then* $q_n \neq 0$ *for* $1 \leqslant n \leqslant N$. *If the above conditions hold for arbitrary* $N \geqslant 2$, *then* $q_n \neq 0$ *for* $n \geqslant 1$.

Next, we assume that $z_{n+1} \in \partial H_n \cap \partial P_{n+1}$ for some $n \in \mathbf{N}$. Then (9) implies $\operatorname{Re} e^{i\varphi_n}z_{n+1}c_{n+1} + |c_{n+1}z_{n+1}| = 0$. Hence, $\operatorname{Im} e^{i\varphi_n}z_{n+1}c_{n+1} = 0$ and, from (6), we obtain

$$(11)\qquad z_{n+1} = -e^{-i(\varphi_n + \varphi_{n+1} + \psi_{n+1})}|z_{n+1}|.$$

This and $z_{n+1} \in \partial H_n$ imply

$$(12)\qquad \operatorname{Re} e^{i\varphi_n}d_n = -\operatorname{Re} e^{i\varphi_n}z_{n+1} = |z_{n+1}|\cos(\varphi_{n+1} + \psi_{n+1}).$$

Hence, $\cos(\varphi_{n+1} + \psi_{n+1}) > 0$ since $\operatorname{Re} e^{i\varphi_n}d_n > 0$. If $|z_{n+1}|$ from (12) is substituted into (11), we obtain

$$(13)\quad z_{n+1} = z'_{n+1} := \frac{-e^{-i(\varphi_n + \varphi_{n+1} + \psi_{n+1})}\operatorname{Re} e^{i\varphi_n}d_n}{\cos(\varphi_{n+1} + \psi_{n+1})}, \qquad \cos(\varphi_{n+1} + \psi_{n+1}) > 0.$$

($z'_{n+1} = -d_n$ if $\varphi_n = \varphi_{n+1} = 0$, $a_{n+1} > 0$, $d_n > 0$.)

Conversely, one verifies that $z'_{n+1} \in \partial H_n \cap \partial P_{n+1}$ holds. Therefore, ∂H_n and ∂P_{n+1} always have exactly one point, z'_{n+1}, in common, provided

$$\cos(\varphi_{n+1} + \psi_{n+1}) > 0.$$

If, in particular, $-b_\nu \notin H_\nu$, $z_\nu \in P_\nu$, $\nu \geqslant 2$, and $-b_1 \in \partial H_1$, then $w_n = \infty$ or $s_2 \circ \cdots \circ s_n(z_{n+1}) = -b_1$ can occur for some $n \geqslant 2$ only if $-b_1 \in \partial D_2$ also. By the above considerations this is only possible if $z_{n+1} = z'_{n+1}$ in (13) and if all disks

$$D_2 = s_2(H_2) \supset s_2 \circ s_3(H_3) \supset \cdots \supset s_2 \circ \cdots \circ s_n(H_n)$$

touch ∂H_1 at the point $-b_1$.

Finally, we assume that $z_n = z$, $H_n = H := \{\zeta \in \mathbf{C}\colon \operatorname{Re} e^{i\varphi}(\zeta + d) \geq 0\}$, $n \geq 1$, with fixed $\varphi \in \mathbf{R}$, $d \in \mathbf{C}$ and $-b_1 \in \partial H$, $-b_\nu \notin H$, $\nu \geq 2$. If for some $n \geq 2$ and $z \in \bigcap_{\nu=2}^{n+1} P_\nu$, $s_2 \circ \cdots \circ s_n(z) = -b_1$ holds, then $-b_1 \in \partial D_2$ also and, hence, $z = z'_{n+1} \in \partial H$. Now $-b_1 \in \partial H \cap \partial D_2$ is equivalent to $\operatorname{Re} e^{i\varphi}(b_1 - d) = 0$ and $|b_1 + z - zc_2| = |zc_2|$. If $\operatorname{Im} e^{i\varphi}(b_1 + z - zc_2) \neq 0$, then

$$|b_1 + z - zc_2| > \operatorname{Re} e^{i\varphi}(b_1 + z - zc_2) = -\operatorname{Re} e^{i\varphi} zc_2 = |zc_2|$$

because $\operatorname{Re} e^{i\varphi}(b_1 + z) = \operatorname{Re} e^{i\varphi}(d + z) = 0$ and (9), since $z \in \partial H \cap \partial P_2$. But this contradicts $-b_1 \in \partial D_2$. Hence, $\operatorname{Im} e^{i\varphi}(b_1 + z - zc_2) = 0$. This yields $\operatorname{Im} e^{i\varphi}(b_1 + z) = 0$, since $-\operatorname{Re} e^{i\varphi} zc_2 = |zc_2|$ implies $\operatorname{Im} e^{i\varphi} zc_2 = 0$. Therefore $b_1 + z = 0$. If $n = 2$, then this, $0 = b_1 + s_2(z)$, and $z \neq 0$ imply $b_2 + z = \infty$. If $n \geq 3$, then $b_1 + z = 0$, $z \neq 0$, and $0 = b_1 + s_2 \circ \cdots \circ s_n(z)$ imply $s_3 \circ \cdots \circ s_n(z) = \infty$. But this is impossible by applying Theorem 1 to the denominators of the continued fraction

$$\frac{1}{b_3 + z} - \frac{za_4}{b_4 + z} - \cdots,$$

since $-b_n \notin H$ for $n \geq 2$. Hence, $q_n \neq 0$ for $n \geq 2$ and $q_1 = 0$ iff $z = -b_1$. We thus have proved

THEOREM 2. (1) *If (cf.* (1), (5), (9), (13)) $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) = 0$ (*i.e.* $-b_1 \in \partial H_1$), $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$, $2 \leq n \leq N$ *and if* $z_n \in P_n$, $2 \leq n \leq N + 1$, *then* $q_N \neq 0$ *provided* $z_{N+1} \neq z'_{N+1}$ *in case* $|b_1 + z_2 - z_2c_2| = |z_2c_2|$ (*i.e.* $-b_1 \in \partial D_2$).

(2) *If* $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) = 0$, $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) > 0$, $n \geq 2$ *and* $z_n \in P_n$, $n \geq 2$, *then* $q_n \neq 0$ *for* $n \geq 2$ *provided* $z_{n+1} \neq z'_{n+1}$, $n \geq 2$, *in case* $|b_1 + z_2 - z_2c_2| = |z_2c_2|$.

(3) *Assume that all* $z_n = z$ *and that for fixed* $\varphi \in \mathbf{R}$, $d \in \mathbf{C}$, $\operatorname{Re} e^{i\varphi} d > 0$ *holds. If* $\operatorname{Re} e^{i\varphi}(b_1 - d) \geq 0$, *and* $\operatorname{Re} e^{i\varphi}(b_n - d) > 0$, $2 \leq n \leq N$, *then* $q_n(z) \neq 0$ *for* $z \in \bigcap_{\nu=2}^{N+1} P_\nu$ *and* $2 \leq n \leq N$. ($q_1(z) = 0$ *iff* $z = -b_1$.) *If the above conditions hold for arbitrary* $N \geq 2$, *then* $q_n(z) \neq 0$ *for* $z \in \bigcap_{\nu=2}^{\infty} P_\nu$ *and* $n \geq 2$.

We now choose $h(c_n) = \operatorname{Re} c_n$ in (10). Then (10) is equivalent to

$$P_n = \left\{\zeta \in \mathbf{C}\colon |\zeta| \leq \operatorname{Re}\left(e^{i\varphi_{n-1}}\zeta \bar{c}_n/|c_n|\right) + 2\operatorname{Re}\left(e^{i\varphi_{n-1}} d_{n-1}\right)\operatorname{Re} c_n/|c_n|\right\}.$$

By (1) and (6), $\bar{c}_n/|c_n| = e^{-i(\varphi_n + \psi_n)}$. Hence,

$$h(c_n) = \operatorname{Re} c_n > 0$$

iff $\cos(\varphi_n + \psi_n) > 0$, and $\operatorname{Re} c_n \leq \frac{1}{2}$ iff $\operatorname{Re} e^{i\varphi_n}(b_n - a_n - d_n) \geq 0$, $n \geq 2$. If these conditions are satisfied, then $\operatorname{Re} e^{i\varphi_n}(b_n - d_n) \geq \operatorname{Re} e^{i\varphi_n} a_n > 0$ holds for $n \geq 2$ and, hence, Theorems 1 and 2(1)(2) yield

COROLLARY 1. *Assume that* $\varphi_n \in \mathbf{R}$, $d_n \in \mathbf{C}$ *such that* $\operatorname{Re} e^{i\varphi_n} d_n > 0$, $n \in \mathbf{N}$. *Let* P_n, $n \geq 2$, *be the parabolic region*

$$P_n = \left\{\zeta \in \mathbf{C}\colon |\zeta| \leq \operatorname{Re} e^{i(\varphi_{n-1} - \varphi_n - \psi_n)}\zeta + 2\cos(\varphi_n + \psi_n)\operatorname{Re} e^{i\varphi_{n-1}} d_{n-1}\right\},$$

where $\cos(\varphi_n + \psi_n) > 0$, $n \geq 2$.

(1) *If* $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) > 0$, $\operatorname{Re} e^{i\varphi_n}(b_n - a_n - d_n) \geq 0$, $2 \leq n \leq N$, *and* $z_n \in P_n$, $2 \leq n \leq N + 1$, *or if, more generally*,

$$|b_1 + z_2 - z_2c_2| > |z_2c_2|,$$

$\operatorname{Re} e^{i\varphi_n}(b_n - a_n - d_n) \geqslant 0$, $2 \leqslant n \leqslant N$ *and* $z_n \in P_n$, $3 \leqslant n \leqslant N+1$, *then* $q_n \neq 0$ *for* $1 \leqslant n \leqslant N$. *If the above conditions hold for all* $N \geqslant 2$, *then* $q_n \neq 0$ *for* $n \geqslant 1$.

(2) *If* $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) = 0$, $\operatorname{Re} e^{i\varphi_n}(b_n - a_n - d_n) \geqslant 0$, $2 \leqslant n \leqslant N$ *and* $z_n \in P_n$, $2 \leqslant n \leqslant N+1$, *then* $q_N \neq 0$ *provided* $z_{N+1} \neq z'_{N+1}$ *in case* $|b_1 + z_2 - z_2c_2| = |z_2c_2|$.

(3) *If* $\operatorname{Re} e^{i\varphi_1}(b_1 - d_1) = 0$, $\operatorname{Re} e^{i\varphi_n}(b_n - a_n - d_n) \geqslant 0$, $n \geqslant 2$, *and* $z_n \in P_n$, $n \geqslant 2$, *then* $q_n \neq 0$ *for* $n \geqslant 2$ *provided* $z_{n+1} \neq z'_{n+1}$, $n \geqslant 2$ *in case* $|b_1 + z_2 - z_2c_2| = |z_2c_2|$.

In particular, we obtain from Theorem 2(3):

COROLLARY 2. *Assume that in* (1) $a_n = |a_n| e^{i\psi}$ *and* $b_n - a_n = |b_n - a_n| e^{i\chi}$ *with fixed* $\psi, \chi \in \mathbf{R}$ *for* $n \geqslant 2$. *Assume also that* $\rho := \inf_{n\geqslant 2} |b_n - a_n| > 0$ *and put* $d := \rho e^{i\chi}$. *If for some* $\varphi \in \mathbf{R}$, $\operatorname{Re} e^{i\varphi}(b_1 - d) \geqslant 0$, $\cos(\varphi+\psi) > 0$ *and* $\cos(\varphi+\chi) > 0$ (*i.e.* $\operatorname{Re} e^{i\varphi} d > 0$), *then the polynomials* $q_n(z)$, *defined by*

$$q_n(z) = (b_n + z)q_{n-1}(z) - a_n z q_{n-2}(z), \qquad n \in \mathbf{N},\ q_{-1} := 0,\ q_0 := 1,$$

have no zeros for $n \geqslant 2$ *in the closed parabolic region*

$$P' = \left\{\zeta \in \mathbf{C}\colon |\zeta| \leqslant \operatorname{Re} e^{-i\psi}\zeta + 2\rho\cos(\varphi+\psi)\cos(\varphi+\chi)\right\}.$$

$q_1(z) = 0$ *iff* $z = -b_1$. *If* $\varphi = -(\chi+\psi)/2$ *and* $|\chi - \psi| < \pi$, *then* $q_n(z)$, $n \geqslant 2$, *have no zeros in the closed parabolic region*

$$P = \left\{\zeta \in \mathbf{C}\colon |\zeta| \leqslant \operatorname{Re} e^{-i\psi}\zeta + 2\rho\cos^2((\chi-\psi)/2)\right\} \supset P'. \tag{14}$$

PROOF. Since $\operatorname{Re} e^{i\varphi}a_n = |a_n|\cos(\varphi+\psi) > 0$ and $\operatorname{Re} e^{i\varphi}d = \rho\cos(\varphi+\chi) > 0$, Corollary 2 follows from Theorem 2(3). Observe that

$$2\cos(\varphi+\psi)\cos(\varphi+\chi) = \cos(2\varphi+\psi+\chi) + \cos(\psi-\chi)$$

is maximal for $\varphi = -(\chi+\psi)/2$. Then

$$\cos(\varphi+\psi) = \cos(\varphi+\chi) = \cos((\chi-\psi)/2) > 0 \quad \text{if } |\chi-\psi| < \pi.$$

4. Application to generalized Bessel polynomials. These polynomials are defined for $n \in \mathbf{N}$ by (cf. [**2**, **6**])

$$Y_n^{(\delta)}(z) := 1 + \sum_{j=1}^{n} \binom{n}{j}(n+\delta+1)\cdots(n+\delta+j)(-z/2)^j, \quad \text{where } \delta \in \mathbf{C}.$$

Then for fixed $m \in \mathbf{N}$ the polynomials $q_n^{(m+\delta)}(z) := z^n Y_n^{(m+\delta-n)}(-2/z)$ satisfy

$$q_n^{(m+\delta)}(z) = (n+m+\delta+z)q_{n-1}^{(m+\delta)}(z) - (n-1)zq_{n-2}^{(m+\delta)}(z)$$
$$\text{with } q_{-1}^{(m+\delta)} := 0,\ q_0^{(m+\delta)} := 1.$$

Hence $q_n^{(m+\delta)}(z)$, $n \in \mathbf{N}$, is of type (1) with $b_n = n+m+\delta$, $a_n = n-1$, $n \in \mathbf{N}$, and, therefore, $b_n - a_n = m+1+\delta$ is independent of n. With the notation of Corollary 2, we have, in this example,

$$\psi = 0, \quad \rho = |m+1+\delta| \quad \text{and} \quad d = m+1+\delta = |m+1+\delta| e^{i\chi} = b_1.$$

If $\varphi = -\chi/2$, then (14) yields a *closed* parabolic region (containing $-b_1$)

$$P = \left\{\zeta \in \mathbf{C}\colon |\zeta| \leqslant \operatorname{Re}\zeta + 2|m+1+\delta|\cos^2(\chi/2)\right\}$$

such that $q_n^{(m+\delta)}(z) \neq 0$ for $n \geqslant 2$, $z \in P$, provided $m + 1 + \delta \neq 0$ and $|\chi| < \pi$. For $0 < |\chi| < \pi/2$, $2\cos^2(\chi/2) = 1 + \cos\chi > 2\cos\chi$ holds and, therefore, P properly contains the parabolic region $P' = \{\zeta \in \mathbf{C}: |\zeta| \leqslant \operatorname{Re}\zeta + 2(m + 1 + \operatorname{Re}\delta)\}$.

In particular, for $n = m \geqslant 2$, this shows that $Y_m^{(\delta)}(z) \neq 0$ for $-2/z \in P$. Thus we have proved

THEOREM 3. *If* $m + 1 + \delta = |m + 1 + \delta| e^{i\chi} \neq 0$ *and* $|\chi| < \pi$, *then all zeros of* $Y_m^{(\delta)}(z)$, $m \geqslant 2$, *are contained in the open cardioid region*

$$C = \left\{z = re^{i\vartheta} \in \mathbf{C}: 0 < r < (1 + \cos\vartheta)/|m + 1 + \delta|\cos^2(\chi/2), |\vartheta| < \pi\right\}.$$

i.e. if $m + \delta + 1$ *lies on the parabola* $|z| + \operatorname{Re} z = 2c > 0$, *then all zeros of* $Y_m^{(\delta)}(z)$, $m \geqslant 2$, *are contained in*

$$C = \left\{z = re^{i\vartheta} \in \mathbf{C}: 0 < r < (1 + \cos\vartheta)/c, |\vartheta| < \pi\right\}.$$

If $0 < |\chi| < \pi/2$, *then* C *is properly contained in the open cardioid region*

$$C' = \left\{z = re^{i\vartheta} \in \mathbf{C}: 0 < r < (1 + \cos\vartheta)/(m + 1 + \operatorname{Re}\delta), |\vartheta| < \pi\right\}.$$

$Y_1^{(\delta)}(z) = 0$ *iff* $z = 2/(2 + \delta)$.

In the special case where $m + \delta + 1$ is real and > 0, M. G. de Bruin, E. B. Saff and R. S. Varga have proved in **[1, 5, 7]** that the result of Theorem 3 is sharp in the sense that each boundary point of $\{z = re^{i\vartheta} \in \mathbf{C}: 0 < r < (1 + \cos\vartheta)/2, |\vartheta| < \pi\}$ is an accumulation point of zeros of the normalized Bessel polynomials

$$Y_m^{(\delta)}(2z/(m + \delta + 1)), \qquad m \in \mathbf{N}, \quad m + \delta + 1 > 0.$$

Whether for *nonreal* $m + \delta + 1$, Theorem 3 (and for *nonreal* a_n, b_n, Corollary 2) still is sharp is an open question. For other new results concerning the location of zeros of polynomials satisfying a three-term recurrence relation see also **[4]**.

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