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REMARK ON LOMONOSOV'S LEMMA

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ABSTRACT. The famous lemma of V. I. Lomonosov states that if $\mathfrak{A}$ is a transitive algebra of operators acting on a complex, infinite-dimensional Banach space X and K is a nonzero compact operator on X, then there is an $A \in \mathfrak{A}$ such that 1 is an eigenvalue of AK. Lomonosov's proof uses Schauder's fixed point theorem. A proof, using only elementary techniques, is given for the case where K has finite-rank.

Let X be a complex, infinite-dimensional Banach space and let $\mathfrak{A}$ be a transitive algebra of operators on X. What is now widely known as Lomonosov's Lemma [**3**, **7**, p. 156; **6**] asserts that for any nonzero compact operator K acting on X, there is an $A \in \mathfrak{A}$ such that 1 is an eigenvalue of AK. Lomonosov's proof uses Schauder's fixed point theorem in an ingenious way. It follows almost immediately from this lemma that every nonscalar operator A on X which commutes with a nonzero compact operator has a nontrivial hyperinvariant subspace [**7**, p. 158; **6**]. In [**6**], the authors remark that a proof of the latter which does not use Schauder's theorem remains to be found. Such a proof has not yet appeared, although H. M. Hilden has succeeded in finding one for the case where A is compact [**7**, p. 158; **6**, **4**]. For some other recent advances employing Lomonosov's results and further reference see the expository paper [**8**], and [**1**, **2**, **5**]. Can a proof of Lomonosov's basic lemma be found which does not use Schauder's fixed point theorem? The following is proved using only elementary techniques.

THEOREM. *Let $\mathfrak{A}$ be a transitive algebra of operators acting on a complex, infinite-dimensional Banach space X. For any nonzero finite-rank operator K on X there is an $A \in \mathfrak{A}$ such that* 1 *is an eigenvalue of AK.*

Two observations are basic to the proof. Firstly, there can be no nonzero operator R on X such that $RBR = 0$ for every $B \in \mathfrak{A}$. Otherwise, for every nonzero vector f in the range of R, either the one-dimensional subspace spanned by f is in the kernel of $\mathfrak{A}$ and hence is invariant under $\mathfrak{A}$, or the closure of $\mathfrak{A}f = \{Af: A \in \mathfrak{A}\}$ is a nontrivial invariant subspace of $\mathfrak{A}$. Secondly we have the following.

LEMMA. *Let $\mathfrak{M}$ be a family of complex matrices for which there exists an integer $k > 1$ such that $C^k = 0$ for every $C \in \mathfrak{M}$. Also, let $\mathfrak{M}$ be a linear manifold (that is, $A, B \in \mathfrak{M}$ and $\lambda \in \mathbf{C}$ implies $A + B$, $\lambda A \in \mathfrak{M}$). Then $A^{k-1}BA^{k-1} = 0$ for every $A, B \in \mathfrak{M}$.*

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PROOF. Let $A, B \in \mathfrak{M}$. Then $(A + B)^k$ is the sum of 2^k terms which can be grouped so that we have $(A + B)^k = 0 = \Sigma_0 + \Sigma_1 + \cdots + \Sigma_k$ with Σ_j the sum of terms of degree j in B ($\Sigma_0 = A^k, \Sigma_1 = BA^{k-1} + ABA^{k-2} + \cdots + A^{k-1}B, \ldots, \Sigma_k = B^k$). Let $z \in \mathbf{C}$. Replacing B by zB in the above gives $0 = \Sigma_0 + z\Sigma_1 + z^2\Sigma_2 + \cdots + z^k\Sigma_k$. It follows from the Fundamental Theorem of Algebra that $\Sigma_j = 0$ ($j = 0, 1, \ldots, k$). In particular, $\Sigma_1 = 0$ so $A^{k-1}\Sigma_1 = 0 = A^{k-1}BA^{k-1}$.

PROOF OF THE THEOREM. Let $\mathfrak{A}$ be transitive, let K have rank $n \geqslant 1$ but suppose there is no $A \in \mathfrak{A}$ such that 1 is an eigenvalue of AK. Then the spectrum $\sigma(AK)$ is $\{0\}$ for every $A \in \mathfrak{A}$. For $A \in \mathfrak{A}$ let T_A be the restriction of KA to the range of K. If $\lambda \in \sigma(T_A)$, then $T_A x = \lambda x$ for some $x \neq 0$ and $AT_A x = AK(Ax) = \lambda(Ax)$. From this it follows that $\sigma(T_A) = \{0\}$ so $T_A^n = 0$, for every $A \in \mathfrak{A}$. Let k be the least positive integer with the property that $T_A^k = 0$ for every $A \in \mathfrak{A}$. If $T_A = 0$ for every $A \in \mathfrak{A}$, then $KAK = 0$ for every $A \in \mathfrak{A}$ and this contradicts the transitivity of $\mathfrak{A}$ (see earlier observation). Thus $k > 1$. The set of transformations $\{T_A : A \in \mathfrak{A}\}$ is closed under addition and scalar multiplication (indeed $T_A + T_B = T_{A+B}$ and $\lambda T_A = T_{\lambda A}$) so, by the lemma, $T_A^{k-1}T_BT_A^{k-1} = 0$ for every $A, B \in \mathfrak{A}$. By the definition of k, $T_{A_0}^{k-1} \neq 0$ for some $A_0 \in \mathfrak{A}$. The operator $R = (KA_0)^{k-1}K$ has the same range as $T_{A_0}^{k-1}$ so $R \neq 0$. But $T_{A_0}^{k-1}T_BT_{A_0}^{k-1} = 0$ gives $RBR = 0$ for every $B \in \mathfrak{A}$, and the latter contradicts the transitivity of $\mathfrak{A}$. This completes the proof.

Admittedly, it is difficult to see how this proof can be modified when K has infinite rank. Is there an infinite-dimensional version of our lemma, concerning linear manifolds of quasinilpotent compact operators on complex Banach spaces? Such a result may lead, via Hilden's techniques, to a proof of Lomonosov's Lemma without recourse to Schauder's Theorem.

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