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ON THE DEGREE OF APPROXIMATION OF A CLASS OF FUNCTIONS BY MEANS OF FOURIER SERIES

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ABSTRACT. In this paper degree of approximation of Lebesgue integrable functions by means of Fourier series is examined.

1. Let f be a periodic function with period 2π and integrable in the sense of Lebesgue. Let

$$f(x) \sim \frac{a_0}{2} + \sum_1^{\infty} (a_k \cos kx + b_k \sin kx).$$

We write

$$\phi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t) - 2f(x)\} \quad \text{and} \quad \Phi(t) = \int_0^t |\phi(u)|\,du.$$

Let

$$w_1(\delta) = w_1[f, x](\delta) = \sup_{|h| \leq \delta} \left\{ \frac{1}{2h} \int_{-h}^{h} |f(x+u) - f(x)|\,du \right\}.$$

It is clear that for $f \in C^*[0, 2\pi]$, $w_1(\delta) \leq w[f](\delta)$, where $w[f](\delta)$ denotes the modulus of continuity of f.

Let $\Lambda = (\lambda_{n,k})$, $k = 0, 1, 2, \ldots, n$, be a triangular matrix and let

$$\sigma_n = \sum_{k=0}^{n} \lambda_{n,k} s_k,$$

where $\{s_k\}$ is a given sequence of numbers. σ_n is called nth Λ-means of $\{s_n\}$. If $\sigma_n \to s$ as $n \to \infty$, we say that $\{s_n\}$ is summable (Λ) to s. We suppose that $\{\lambda_{n,k}\}$ is nonnegative with $\sum_{k=0}^{n} \lambda_{n,k} = 1$, $n = 0, 1, \ldots$. Then a necessary and sufficient condition for regularity of the Λ-method is $\lim_{n\to\infty} \lambda_{n,k} = 0$ for each k.

For

$$\lambda_{n,k} = \frac{p_{n-k}}{P_n}, \qquad P_n = p_0 + p_1 + \cdots + p_n, p_0 > 0,$$

the Λ-method reduces to the (N, p_n) method. Similarly for $\lambda_{n,k} = p_k/P_n$, we get $(\overline{N}, p_n)$ means.

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In what follows we assume that C is a positive constant not necessarily the same at each occurrence.

2. In this paper we establish the following

THEOREM. *Suppose for fixed* x, $w_1(\delta) < \infty$ *for* $\delta \in (0, \pi]$, *and let* $\sigma_n(x)$ *denote the nth* Λ*-means of the Fourier series of* $f(x)$. *If* $\{\lambda_{n,k}\}$ *is nondecreasing with respect to* k, *then*

$$|\sigma_n(x) - f(x)| \leqslant C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}. \tag{2.1}$$

3. PROOF. We have

$$\begin{aligned}\sigma_n(x) - f(x) &= \frac{2}{\pi}\int_0^{\pi} \phi(t) \sum_{k=0}^{n} \lambda_{n,k} D_k(t)\, dt \\ &= \frac{2}{\pi}\left(\int_0^{\pi/n+1} + \int_{\pi/n+1}^{\pi}\right) = I_1 + I_2, \quad \text{say,}\end{aligned}$$

where

$$D_k(t) = \frac{\sin\left(k+\frac{1}{2}\right)t}{2\sin t/2}.$$

Since

$$\Phi(t) = \int_0^t |\phi(u)|\,du \leqslant \frac{1}{2}\int_{-t}^{t} |f(x+u) - f(x)|\,du \leqslant t w_1(t),$$

it follows that

$$\begin{aligned}|I_1| &\leqslant \frac{2}{\pi}\int_0^{\pi/n+1} |\phi(t)| \sum_{k=0}^{n} \lambda_{n,k}\left(k+\frac{1}{2}\right) dt \leqslant \frac{2(n+1)}{\pi}\int_0^{\pi/n+1} |\phi(t)|\,dt \\ &\leqslant 2w_1\left(\frac{\pi}{n+1}\right) \leqslant 2\sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.\end{aligned} \tag{3.1}$$

Let γ_n be a sequence of linear functions on $[k, k+1]$ such that $\gamma_n(k) = \lambda_{n,n-k}$, $k = 0, 1, 2, \ldots, n$, and let $F_n(t) = \int_0^t \gamma_n(u)\,du$, $t \geqslant 0$. Then

$$\begin{aligned}F_n(k) &= \sum_{\nu=0}^{k-1} \frac{\gamma_n(\nu+1) + \gamma_n(\nu)}{2} = \sum_{\nu=0}^{k-1} \frac{\lambda_{n,n-\nu-1} + \lambda_{n,n-\nu}}{2} \\ &\leqslant \sum_{\nu=0}^{k} \lambda_{n,n-\nu} \leqslant 2F_n(k).\end{aligned}$$

Using the well-known estimate of McFadden [5],

$$\left|\sum_{k=a}^{b} \lambda_{n,n-k} e^{i(n-k)t}\right| \leqslant 2(2\pi+1) F_n\left(\frac{\pi}{t}\right),$$

where $0 \leqslant a \leqslant b \leqslant \infty$, $0 < t \leqslant \pi$ and n is any nonnegative integer, we have

$$|I_2| \leqslant \frac{2}{\pi} \int_{\pi/n+1}^{\pi} |\phi(t)| \left| \sum_{k=0}^{n} \lambda_{n,k} D_k(t) \right| dt \leqslant C \int_{\pi/n+1}^{\pi} \left| \frac{\phi(t)}{t} \right| F_n\left(\frac{\pi}{t}\right) dt$$

$$= C\left\{ \left[\frac{\Phi(t)}{t} F_n\left(\frac{\pi}{t}\right) \right]_{\pi/n+1}^{\pi} + \int_{\pi/n+1}^{\pi} \frac{\Phi(t)}{t^2} F_n\left(\frac{\pi}{t}\right) dt \right.$$

$$\left. + \int_{\pi/n+1}^{\pi} \frac{\Phi(t)}{t} F_n'\left(\frac{\pi}{t}\right) \cdot \frac{\pi}{t^2} dt \right\}$$

$$= C\left\{ \frac{\Phi(\pi)}{\pi} F_n(1) - \frac{(n+1)}{\pi} \Phi\left(\frac{\pi}{n+1}\right) F_n(n+1) \right.$$

$$\left. + \int_1^{n+1} \frac{\Phi(\pi/t)}{\pi} F_n(t)\, dt + \int_1^{n+1} \frac{t}{\pi} \Phi\left(\frac{\pi}{t}\right) F_n'(t)\, dt \right\}$$

$$\leqslant C w_1(\pi) \lambda_{n,n} + C \sum_{k=1}^{n} \int_k^{k+1} \frac{\pi}{t} w_1\left(\frac{\pi}{t}\right) F_n(t)\, dt + C \sum_{k=1}^{n} \int_k^{k+1} w_1\left(\frac{\pi}{t}\right) \gamma_n(t)\, dt$$

$$= I_{21} + I_{22} + I_{23}, \quad \text{say.}$$

Obviously,

$$I_{21} \leqslant C \sum_{k=0}^{n} w_1\left(\frac{\pi}{k+1}\right) \lambda_{n,n-k} \leqslant C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}$$

and

$$I_{22} \leqslant C \sum_{k=1}^{n} \frac{w_1(\pi/k)}{k} F_n(k+1) \leqslant C \sum_{k=0}^{n-1} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}$$

$$\leqslant C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.$$

Similarly,

$$I_{23} \leqslant C \sum_{k=1}^{n} w_1\left(\frac{\pi}{k}\right) \left(\frac{\gamma_n(k) + \gamma_n(k+1)}{2} \right) \leqslant C \sum_{k=1}^{n} w_1\left(\frac{\pi}{k}\right) \lambda_{n,n-k}$$

$$\leqslant C \sum_{k=0}^{n} w_1\left(\frac{\pi}{k+1}\right) \lambda_{n,n-k} \leqslant C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}.$$

Thus

$$|I_2| \leqslant C \sum_{k=0}^{n} \frac{w_1(\pi/(k+1))}{k+1} \sum_{\nu=0}^{k} \lambda_{n,n-\nu}. \tag{3.2}$$

From (3.1) and (3.2) the proof of our theorem follows.

4. Taking $\lambda_{n,k} = p_{n-k}/P_n$, where $\{p_k\}$ is a positive nonincreasing sequence, we deduce from (2.1) the following theorem due to Markiewicz **[4]**; the case $p_n = 1$ is due to Aljančić, Bojanic and Tomić **[1]**.

COROLLARY 1. *If for fixed* x, $w_1(\delta) < \infty$ *for* $\delta \in (0, \pi]$, *then*

$$|t_n(x) - f(x)| \leq \frac{C}{P_n} \sum_{k=0}^{n} \frac{P_k}{k+1} w_1\left(\frac{\pi}{k+1}\right),$$

where $t_n(x)$ *denotes the* (N, p_n) *means of the Fourier series of* $f(x)$.

This result in weaker form, where w_1 is replaced by w, is due to Holland, Sahney and Tzimbalario [3]. For related results concerning Cesàro summability see Obrechkoff [6] and Flett [2].

COROLLARY 2. *If* $\{\lambda_{n,k}\}$ *is a nonincreasing sequence with respect to* k, *then under the hypothesis of the theorem*,

$$|\sigma_n(x) - f(x)| \leq C \sum_{k=0}^{n} \lambda_{n,k} w_1\left(\frac{\pi}{k+1}\right).$$

PROOF. Let $t_n^*(x)$ denote the $(C, 1)$ mean of the Fourier series. Then taking $\lambda_{n,k} = (n+1)^{-1}$ in our Theorem, we have

$$|t_n^*(x) - f(x)| \leq \frac{C}{n+1} \sum_{k=0}^{n} w_1\left(\frac{\pi}{k+1}\right). \tag{4.1}$$

Using a partial summation formula of Abel,

$$\begin{aligned}\sigma_n(x) - f(x) &= \sum_{\nu=0}^{n} \lambda_{n,\nu}(s_\nu(x) - f(x)) \\ &= \sum_{\nu=0}^{n-1} \Delta\lambda_{n,\nu}(\nu+1)(t_\nu^*(x) - f(x)) + \lambda_{n,n}(n+1)(t_n^*(x) - f(x)) \\ &= \sum_{\nu=0}^{n} (\nu+1)(t_\nu^*(x) - f(x))\Delta\lambda_{n,\nu}.\end{aligned}$$

Since $\Delta\lambda_{n,\nu} \geq 0$ we have, from (4.1),

$$\begin{aligned}|\sigma_n(x) - f(x)| &\leq C \sum_{\nu=0}^{n} \Delta\lambda_{n,\nu} \sum_{k=0}^{\nu} w_1\left(\frac{\pi}{k+1}\right) \\ &= C \sum_{k=0}^{n} w_1\left(\frac{\pi}{k+1}\right) \sum_{\nu=k}^{n} \Delta\lambda_{n,\nu} = C \sum_{k=0}^{n} \lambda_{n,k} w_1\left(\frac{\pi}{k+1}\right).\end{aligned}$$

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