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# ON THE NECESSITY OF THE HÖRMANDER CONDITION FOR MULTIPLIERS ON $\mathbf{H}^p(\mathbf{R}^n)$

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ABSTRACT. In this paper we prove that a class of multiplier operators on $\mathbf{H}^p(\mathbf{R}^n)$, that send atoms to molecules boundedly, must satisfy a Hörmander condition. This provides a partial converse to a theorem of Taibleson and Weiss.

In [1], Coifman and Weiss began the systematic study of the atomic and molecular structure of Hardy spaces on spaces of homogeneous type. This line of study was continued by Taibleson and Weiss [4] where they provide the details for the molecular decomposition of $\mathbf{H}^p$-spaces associated with $\mathbf{R}^n$ and the unit disk, and show how these decompositions can be used to obtain multiplier theorems. In this paper we will provide a converse to the main multiplier theorem of Taibleson and Weiss for a certain class of multipliers.

Let us begin by introducing some notation. We will assume that the reader is familiar with the basic properties of $\mathbf{H}^p(\mathbf{R}^n)$, $0 < p \leqslant 1$, and its atomic and molecular decompositions. If $S$ denotes a set, then $\xi_S$ will denote the indicator function of $S$. For a multi-index $\beta = (\beta_i)$ of nonnegative integers, $D^\beta$ will denote the differential operator $(\partial/\partial x_1)^{\beta_1}(\partial/\partial x_2)^{\beta_2} \cdots (\partial/\partial x_n)^{\beta_n}$. A bounded multiplier operator $T_m$ on $\mathbf{H}^p(\mathbf{R}^n)$ determined by a bounded function $m$ is defined by $T_m(a) = (m\hat{a})^{\vee}$ and $\|T_m(a)\| < C$ for all atoms $a$. A function $m$ is said to be homogeneous (of degree zero) if $m(rx) = m(x)$ for all $r > 0$, and is said to satisfy a Hörmander condition if

$$(\#) \qquad R^{2|\beta|-n} \int_{R<|x|<2R} |D^\beta m(x)|^2 \, dx \leqslant A$$

for all $R > 0$ and $0 < |\beta| < t$, $t$ an integer. For homogeneous $m$ the Hörmander condition $(\#)$ is equivalent to

$$(\#\#) \qquad \int_{1<|x|<2} |D^\beta m(x)|^2 \, dx \leqslant A.$$

In [4] Taibleson and Weiss show that if $m$ satisfies $(\#)$ for some $t > n/2$, then for $1/p - 1/2 < t/n$, $T_m$ sends $(p, 2, t-1)$-atoms boundedly to $(p, 2, [n(1/p - 1)], t/n - 1/2)$-molecules; and hence, $T_m$ is a bounded operator on $\mathbf{H}^p(\mathbf{R}^n)$. The purpose of this work is to provide a converse to this result for homogeneous and

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related multipliers. We begin with a statement of the theorem:

THEOREM. *Suppose $t$ is the smallest integer greater than $n(1/p - 1/2)$ and $m$ is a bounded homogeneous function. If $T_m$ maps $(p, 2, 0)$ atoms boundedly to $(p, 2, 0, t/n - 1/2)$-molecules, then $m$ satisfies the Hörmander condition $(\#)$ for $|\beta| \leq t$.*

PROOF OF THE THEOREM. The proof will consist of the following lemmas and discussion. Note that it is sufficient to bound $(\#)$ or $(\#\#)$ for $|\beta| = t$. We will fix a $\mathbf{C}^\infty$-test function $\eta$ with the following properties:

(i) $\operatorname{supp}(\eta) \subseteq \{x\colon \frac{1}{2} \leq |x| \leq 4\}$,

(ii) $\eta(z) = 1$ for $z \in \{x\colon 1 \leq |x| \leq 2\}$,

(iii) $0 \leq \eta(z) \leq 1$ for $z \in \mathbf{R}^n$,

(iv) $\eta$ is radial.

Let $M = \check{\eta}$ and we compute

$$(\#) = \int_{1<|y|<2} |D^\beta m(y)|^2\, dx = \int_{1<|y|<2} |(D^\beta m(y))\eta(y)|^2\, dy$$

$$= \int_{1<|y|<2} |D^\beta(m\eta)(y)|^2\, dy \quad \text{by (ii)}$$

$$= \int_{\mathbf{R}^n} |D^\beta(m\hat{M})(y)|^2\, dy$$

$$= C\int_{\mathbf{R}^n} ||x|^t (m\hat{M})\check{}(x)|^2\, dx \quad \text{as } t = |\beta|$$

$$= C\int_{\mathbf{R}^n} ||x|^t T_m(M)(x)|^2\, dx$$

$$(\mathrm{I}) = C\| |\cdot|^t T_m(M)\|_2^2.$$

To estimate (I) we will express $M$ as a sum of atoms and use the fact that $T_m$ sends atoms to molecules boundedly. To this end, define sets $B_k$ by $B_0 = \{x\colon |x| \leq 1\}$ and $B_k = \{x\colon 2^{k-1} < |x| \leq 2^k\}$ for $k \geq 1$, and define functions $g_k$, $h_k$, and $a_k$ by $g_k = M \cdot \xi_{B_k}$, $h_k = \{|B_k|^{-1}\int_{B_k} M(x)\,dx\} \cdot \xi_{B_k} = \alpha_k |B_k|^{-1}\xi_{B_k}$, and $a_k = g_k - h_k$. Note $M = \Sigma g_k$.

LEMMA 1. *There exists a nonzero constant $C$, independent of $k$, such that $Ca_k$ is a $(p, 2, 0)$-atom for each $k$.*

PROOF. From the definition of $a_k$, it follows directly that $\int a_k = 0$ and each $a_k$ has compact support. Thus, we need only show that, independent of $k$, there is a contant $C$ such that

$$C|B_k|^{1/p-1/2}\|a_k\|_2 \leq 1.$$

Fix $k \in \mathbf{N}$ and let $B = B_k$. Using the fact that $\eta = M$ is a test-function, we have

$$\begin{aligned}
|B|^{1/2}\|a_k\|_2 &= C_1 2^{kn(1/p-1/2)}\left\{\int_B |\eta(x) - C_1^{-1}2^{-kn}\alpha_k|^2\,dx\right\}^{1/2} \\
&\leq C_1 2^{kn(1/p-1/2)}\left\{\int_B |\eta(x)|\,dx\right\}^{1/2} + 2^{-kn/2}|\alpha_k|\,|B|^{1/2} \\
&\leq C_1C_2 2^{kn(1/p-1/2)}\left\{\int_{|x|>2^{k-1}} |x|^{-2n/p}\,dx\right\}^{1/2} + C_1|\alpha_k| \\
&\leq C_3 + C_1|\alpha_k| \leq C^{-1}.
\end{aligned}$$

The last inequality is the result of the observation that

$$|\alpha_k| = \left|\int g_k\right| \leq \int |M| < \infty. \quad \square$$

We will need the following fact.

LEMMA 2. *With $a_k$ and $t$ as defined previously, $\sum 2^{kt}\|a_k\|_2 < \infty$.*

PROOF. Fix $k \in \mathbf{N}$ and let $B = B_k$. Note $t < n$. We compute

$$\begin{aligned}
2^{kt}\|a_k\|_2 \leq 2^{kn}\|a_k\|_2 &= \left\{\int_B 2^{kn}|\check{\eta}(x) - C\alpha_k 2^{-kn}|^2\,dx\right\}^{1/2} \\
&\leq 2^{kn}\left\{\int_B |\check{\eta}(x)|^2\,dx\right\}^{1/2} + C|\alpha_k|\,|B|^{1/2} \\
&\leq 2^{kn}\left\{\int_B |\check{\eta}(x)|^2\,dx\right\}^{1/2} + C^2 2^{kn/2}\int_B |\check{\eta}(x)|\,dx.
\end{aligned}$$

As $\eta$ is a test function, there exists a constant $A$ such that $|x|^{2n}|\eta(x)| < A$. Thus we have

$$2^{kt}\|a_k\|_2 < 2^{kn}\cdot\left\{\int_B A^2|x|^{-4n}\,dx\right\}^{1/2} + C^2 2^{kn/2}\int_B A|x|^{-2n}\,dx < C' 2^{-kn/2}.$$

Thus the sum $\sum 2^{kt}\|a_k\|_2$ is finite. $\square$

We now return to the estimation of (I):

$$\begin{aligned}
\||\cdot|^t T_m(M)\|_2 &= \||\cdot|^t T_m\left(\sum g_k\right)\|_2 = \||\cdot|^t T_m\left(\sum (g_k - h_k) + \sum h_k\right)\|_2 \\
&\leq \||\cdot|^t T_m\left(\sum a_k\right)\|_2 + \||\cdot|^t T_m\left(\sum h_k\right)\|_2 = (\mathrm{II}) + (\mathrm{III}).
\end{aligned}$$

First we estimate (II). From Lemma 1, we have $2^{2kn(1/p-1/2)}\|a_k\|_2^2 < A$ for all $k$, and thus there exists $\gamma_k > 0$ such that $\gamma_k^2 2^{2kn(1/p-1/2)}\|a_k\|_2^2 = 1$. So $\gamma_k a_k$ is a $(p, 2, 0)$-atom and $T_m(\gamma_k a_k)$ is a constant multiple (independent of $k$) of a

$(p, 2, 0, t/n - 1/2)$-molecule by our hypothesis on $T_m$. Thus

$$\begin{aligned}
(\mathrm{II}) &= \| |\cdot|^t T_m\left(\sum a_k\right)\|_2 < \sum \gamma_k^{-1} \| |\cdot|^t T_m(\gamma_k a_k)\|_2 \\
&< \sum \gamma_k^{-1} \| T_m(\gamma_k a_k)\|_2^{1-(t/n)(2p/(2-p))} \\
&< \left\{\int_{\Sigma_{n-1}} |m(x)|^2\, d\sigma(x)\right\}^{(1/2)-(t/n)(2p/(2-p))} C\sum \gamma_k^{-1} \|\gamma_k a_k\|_2^{1-(t/n)(2p/(2-p))} \\
&= C\sum \frac{\|a_k\|_2}{\left(\gamma_k^2 \|a_k\|_2^2\right)^{(t/n)(p/(2-p))}} \\
&= C\sum \|a_k\|_2 \left(2^{2kn((1/p)-(1/2))}\right)^{(t/n)(p/(2-p))} \\
&= C\sum 2^{kt}\|a_k\|_2 .
\end{aligned}$$

From Lemma 2, this last sum is finite. Thus (II) is bounded.

We now turn our attention to (III). Note that

$$\int\left(\sum h_k\right) = \sum \int_{B_k} h_k = \sum \alpha_k = \sum \int_{B_k} M = \int_{\mathbf{R}^n} M = \eta(0) = 0.$$

Thus it may be possible to express $\Sigma h_k$ as a sum of atoms. Let $\xi_k' = |B_k|^{-1}\xi_{B_k}$ and $H_j = \Sigma_{k=j}^{\infty}\alpha_k$. By using summation by parts and the fact that $H_0 = 0$, we have

$$\sum_{k=0}^{\infty} h_k = \sum_{k=0}^{\infty} \alpha_k \xi_k' = \sum_{k=0}^{\infty} (H_k - H_{k+1})\xi_k' = \sum_{k=0}^{\infty} H_{k+1}(\xi_{k+1}' - \xi_k').$$

Now $\int(\xi_{k+1}' - \xi_k') = 0$ for all $k$ and thus $b_k = H_{k+1}(\xi_{k+1}' - \xi_k')$ has integral zero. In fact $b_k$ is a $(1, 2, 0)$-atom (up to a constant independent of $k$) for

$$|\operatorname{supp}(b_k)| \int |b_k(x)|^2\, dx \leq 2^{kn} C \int |b_k(x)|^2\, dx \leq 2^{kn} C\left(\sum_{j=k}^{\infty} |\alpha_j|^2\right) \leq K$$

as $|\alpha_j| \leq C_1 2^{-nj/2}$. We now have $\Sigma h_k$ written as a sum of atoms $b_k$ and the estimation of (III) proceeds exactly as in the estimation of (II). For each $k \in N$, one selects a positive constant $\beta_k$ such that $\beta_k^2 2^{2kn(1/p-1/2)}\|b_k\|_2^2 = 1$ and then continues as before. Putting these inequalities together, we have bounded (I), and consequently also (#). We have now completed the proof of the theorem.

REMARKS. The homogeneity of the multiplier $m$ was used in two equalities:

(1) $\|T_m(a)\|_2 = c\|a\|_2$ for radial $a$ and

(2) $|D^\beta(m_R)(y)| = R^{-|\beta|}|D^\beta(m)(y)|$

where $m_R(z) = m(Rz)$. If a multiplier $m$ satisfies the following inequalities:

(1)′ $c_1\|a\|_2 \leq \|T_m(a)\|_2$ for radial $a$ and

(2)′ $|D^\beta(m_R)(y)| \leq cR^{-|\beta|}|D^\beta(m)(y)|$,

then the results of this paper hold for the operator $T_m$. For example, multipliers with complex homogeneity ($m(rx) = r^{i\gamma}m(x)$ where $r > 0$ and $\gamma \in \mathbf{R}^*$) satisfy inequalities (1)′ and (2)′.

In **[2]**, one finds results analogous to those presented here, but in the context of Hardy spaces associated with local fields.

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