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# SOME INTEGRAL FORMULAS FOR HYPERSURFACES AND A GENERALIZATION OF THE HILBERT-LIEBMANN THEOREM

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ABSTRACT. R. C. Reilly calculated the variations of functions of the mean curvatures for hypersurfaces in Euclidean space. In the present paper, using Reilly's formulas, we derive some general integral formulas for hypersurfaces, which generalize the well-known Minkowski formulas, and then apply those formulas to obtain some characterizations of the hypersphere.

Let $M$ be a closed hypersurface in $E^{n+1}$ and $H_\gamma$ the $r$th mean curvature of $M$, i.e.

$$H_\gamma = \sum \lambda_1 \lambda_2 \cdots \lambda_\gamma, \tag{1}$$

where $\lambda_1, \ldots, \lambda_n$ are the principal curvatures of $M$. R. C. Reilly proved the following formula for the variation [**1**]:

$$\begin{aligned}(2)\quad \frac{d}{dt}\int f(H_1,\ldots,H_n)\,dV\Big|_{t=0} &= \int_M y\Big\{-H_1 f(H_1,\ldots,H_n) \\ &\quad + \sum_{\gamma=1}^{n}\big(H_\gamma H_1 - (\gamma+1)H_{\gamma+1}\big)D_\gamma f(H_1,\ldots,H_n) \\ &\quad + \sum_{\gamma=1}^{n}\big(D_\gamma f(H_1,\ldots,H_n)\big)_{,ij}T^{ij}_{\gamma-1}\Big\}\,dV,\end{aligned}$$

where $f(H_1,\ldots,H_n)$ is any smooth function, $D_\gamma f(H_1,\ldots,H_n) = \partial f(H_1,\ldots,H_n)/\partial H_\gamma$, $T^{ij}_{\gamma-1}$ is the Newton tensor, $y$ is the normal component of the variational vector, and $H_{n+1} \equiv 0$.

Let $X$ be the position vector and consider the deformation

$$M_t\colon X_t = X + tX = (1+t)X. \tag{3}$$

Now we want to compute $(d/dt)\int_M f(H_1(t),\ldots,H_n(t))\,dV(t)\,|_{t=0}$ directly. Choose a local frame field $Xe_1,\ldots,e_n, e_{n+1}$ such that $e_{n+1}$ is the unit normal vector to $M$ at $X$. From (3) we get

$$dX_t = (1+t)\,dX,$$

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so $C_{n+1}$ is also the unit normal vector to $M_t$. Let

$$dX = \omega_i e_i, \quad dX_t = \omega_i(t) e_i, \qquad 1 \leq i \leq n;$$

we get

$$\omega_i(t) = (1+t)\omega_i, \tag{4}$$

$$\omega_{n+1,i}(t) = \omega_{n+1,i}, \tag{5}$$

so

$$dV(t) = (1+t)^n \, dV. \tag{6}$$

For any real parameter $\lambda$ we have

$$\begin{aligned} (7)\quad & (\omega_1(t) + \lambda\omega_{n+1,1}(t)) \wedge (\omega_2(t) + \lambda\omega_{n+1,2}(t)) \wedge \cdots \wedge (\omega_n(t) + \lambda\omega_{n+1,n}(t)) \\ &= \sum_{\gamma=0}^{n} H_\gamma(t)\lambda^\gamma \omega_1(t) \wedge \cdots \wedge \omega_n(t) = \sum_{\gamma=0}^{n} H_\gamma(t)\lambda^\gamma (1+t)^n \omega_1 \wedge \cdots \wedge \omega_n. \end{aligned}$$

On the other hand we have

$$\begin{aligned} (8)\quad & (\omega_1(t) + \lambda\omega_{n+1,1}(t)) \wedge \cdots \wedge (\omega_n(t) + \lambda\omega_{n+1,n}(t)) \\ &= ((1+t)\omega_1 + \lambda\omega_{n+1,1}) \wedge \cdots \wedge ((1+t)\omega_n + \lambda\omega_{n+1,n}) \\ &= (1+t)^n \sum_{\gamma=0}^{n} H_\gamma \left(\frac{\lambda}{1+t}\right)^\gamma \omega_1 \wedge \omega_2 \wedge \cdots \wedge \omega_n. \end{aligned}$$

From (7) and (8) we have

$$H_\gamma(t) = H_\gamma / (1+t)^\gamma. \tag{9}$$

Due to (6) and (9) we get

$$\frac{d}{dt} \int_M f(H_1(t), \ldots, H_\gamma(t)) \, dV(t) \Big|_{t=0} = \int_M \left( nf - \sum_{\gamma=1}^{n} \gamma D_\gamma f \cdot H_\gamma \right) dV. \tag{10}$$

Comparing it with (2) we get

THEOREM 1.

$$\begin{aligned} (11)\quad & \int_M \left( nf - \sum_{\gamma=1}^{n} \gamma D_\gamma f H_\gamma \right) dV \\ &= \int_M \left\{ -H_1 f + \sum_{\gamma=1}^{n} (H_\gamma H_1 - (\gamma+1) H_{\gamma+1}) D_\gamma f + \sum_{\gamma=1}^{n} (D_\gamma f)_{,ij} T_{\gamma-1}^{ij} \right\} P \, dV, \end{aligned}$$

*where $P$ is the support function of $M$.*

Setting $f = H_\gamma$, (11) gives the well-known Minkowski formulas. Using Theorem 1 we can prove

THEOREM 2. *A closed strictly convex hypersurface in $E^{n+1}$ with $H_\gamma = \text{const}$ is a hypersphere.*

PROOF. Let $f = H_\gamma^{n/\gamma}$. (11) gives

$$0 = \int_M \frac{H_\gamma^{(n-\gamma)/\gamma}}{\gamma}\big((n-\gamma)H_1H_\gamma - n(\gamma+1)H_{\gamma+1}\big)P\,dV + \int_M \frac{n}{\gamma}\big(H_\gamma^{(n-\gamma)/\gamma}\big)_{,ij}T^{ij}_{\gamma-1}P\,dV. \tag{12}$$

Due to $H_\gamma = \text{const}$ we have

$$\int_M \frac{H_\gamma^{(n-\gamma)/\gamma}}{\gamma}\big((n-\gamma)H_1H_\gamma - n(\gamma+1)H_{\gamma+1}\big)P\,dV = 0. \tag{13}$$

Choose the origin in $E^{n+1}$ such that $P > 0$; since $M$ is a strictly convex hypersurface we have

$$H_\gamma > 0 \tag{14}$$

and [**2**]

$$(n-\gamma)H_1H_\gamma - n(\gamma+1)H_{\gamma+1} \geqslant 0. \tag{15}$$

The equality sign holds here if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Hence $M$ must be a hypersphere.

This theorem is a generalization of the Hilbert-Liebmann theorem.

Furthermore we get the following result from (12).

THEOREM 3. *Let $M$ be a strictly convex closed hypersurface in $E^{n+1}$ and the origin an interior point of $M$. Then we have*

$$\int_M \big(H_\gamma^{(n-\gamma)/\gamma}\big)_{,ij}T^{ij}_{\gamma-1}P\,dV \leqslant 0. \tag{16}$$

*The equality sign holds if and only if $M$ is a hypersphere.*

THEOREM 4. *A closed strictly convex hypersurface in $E^{n+1}$ with $H_\gamma/H_\mu = \text{const}$ (for fixed $\gamma$, $\mu$), $1 \leqslant \mu < \gamma < n$, is a hypersphere.*

PROOF. In the integral formula (11) we put

$$f = H_\gamma^{(n-\mu)/(\gamma-\mu)}/H_\mu^{(n-\gamma)/(\gamma-\mu)}. \tag{17}$$

Then we get

$$\begin{aligned} 0 = \int_M \frac{1}{\gamma-\mu}\frac{H_\gamma^{(n-\gamma)/(\gamma-\mu)}}{H_\mu^{(n-\mu)/(\gamma-\mu)}}\big((n-\gamma)(1+\mu)H_\gamma H_{\mu+1} \\ - (n-\mu)(1+\gamma)H_\mu H_{\gamma+1}\big)P\,dV \\ + \int_M\left[\frac{n-\mu}{\gamma-\mu}\left(\frac{H_\gamma}{H_\mu}\right)^{(n-\gamma)/(\gamma-\mu)}_{,ij}T^{ij}_{\gamma-1} - \frac{n-\gamma}{\gamma-\mu}\left(\frac{H_\gamma}{H_\mu}\right)^{(n-\mu)/(\gamma-\mu)}_{,ij}T^{ij}_{\mu-1}\right]P\,dV. \end{aligned} \tag{18}$$

Since $H_\gamma/H_\mu = \text{const}$ we have

$$(19)\quad \frac{1}{\gamma-\mu}\int_M \frac{H_\gamma^{(n-\gamma)/(\gamma-\mu)}}{H_\mu^{(n-\mu)/(\gamma-\mu)}}\big((n-\gamma)(1+\mu)H_\gamma H_{\mu+1} - (n-\mu)(1+\gamma)H_\mu H_{\gamma+1}\big)P\,dV = 0.$$

Choose the origin in $E^{n+1}$ such that $P > 0$. For a strictly convex hypersurface the following inequality is valid [**2**]:

$$(20)\quad \frac{H_{\gamma+1}/\binom{n}{\gamma+1}}{H_\gamma/\binom{n}{\gamma}} \leqslant \frac{H_\gamma/\binom{n}{\gamma}}{H_{\gamma-1}/\binom{n}{\gamma-1}} \leqslant \cdots \leqslant \frac{H_{\mu+1}/\binom{n}{\mu+1}}{H_\mu/\binom{n}{\mu}},$$

so

$$(21)\quad (n-\gamma)(1+\mu)H_\gamma H_{\mu+1} - (n-\mu)(1+\gamma)H_\mu H_{\gamma+1} \geqslant 0.$$

The equality sign holds if and only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n$. Hence $M$ must be a hypersphere.

## REFERENCES

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