

A GENERALIZATION OF THE POINCARÉ-BIRKHOFF THEOREM

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ABSTRACT. A generalized form of the Poincaré-Birkhoff theorem is proved. The generalization is useful for the further applications of this famous fixed point theorem.

1. Introduction and statement of the result. “To establish the existence of periodic orbits in the restricted problem of three bodies Poincaré introduced his celebrated ‘last geometric theorem’ but was unable to prove it... A number of mathematicians had attempted to prove this theorem and at least one erroneous proof was published. Urged on by his colleagues, Birkhoff gave a beautiful proof of the theorem that went directly to the heart of the problem. Modified forms of the theorem were also presented by Birkhoff.”—quoted from Marston Morse [3].

Today, this *Poincaré’s last geometric theorem* is known as the Poincaré-Birkhoff theorem. In the following, we will give a statement of this theorem in modern terms.

Let \mathbf{R}^2 be the (x, y) -plane. Let $H: r > 0$ be the half-plane of the (r, θ) -plane. We will regard H as a covering space of $\mathbf{R}^2 \setminus \{O\}$, where O is the origin of \mathbf{R}^2 . Let $P: H \rightarrow \mathbf{R}^2 \setminus \{O\}$ be the usual covering projection: $P(r, \theta) = (r \cos \theta, r \sin \theta)$. Recall that, for any continuous map $F: D \subset \mathbf{R}^2 \setminus \{O\} \rightarrow \mathbf{R}^2 \setminus \{O\}$, a map $\tilde{F}: P^{-1}(D) \rightarrow H$ is called a lifting of F to H , if the following relation holds:

$$P \circ \tilde{F} = F \circ P.$$

Denote by A_0 the annulus: $R_1^2 \leq x^2 + y^2 \leq R_2^2$, $0 < R_1 < R_2$.

POINCARÉ-BIRKHOFF THEOREM. *Let T be an area-preserving homeomorphism of A_0 onto itself such that both boundary circles of A_0 are invariant under T . Suppose that T has a lifting $\tilde{T}: P^{-1}(A_0) \rightarrow P^{-1}(A_0)$ defined by*

$$(1.1) \quad r^* = f(r, \theta), \quad \theta^* = \theta + g(r, \theta)$$

where f and g are continuous and 2π -periodic in θ . If the “twist condition”: $g(R_1, \theta)g(R_2, \theta) < 0$ holds, then \tilde{T} has at least two fixed points such that their images under the projection P are two different fixed points of T .

In this theorem, the condition that T maps A_0 onto itself is a severe restriction for the application of the theorem. This was certainly noticed by Birkhoff who was interested in the dynamical applications of the theorem. He then made efforts to remove this condition. In [2], he succeeded to such an extent that the outer boundary circle of A_0 is permitted to vary under T while the inner boundary circle remains invariant. This result now can be formulated in a more general form through a modification by H. Jacobowitz (see [4]).

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Let A_1 denote an annular region in \mathbf{R}^2 such that its inner boundary curve C_1 is a circle: $x^2 + y^2 = R_1^2$, $R_1 \geq 0$, and the outer boundary curve C_2 is a closed simple curve. Note that C_1 will degenerate into the origin in the case $R_1 = 0$. In this degenerate case we assume that A_1 does not contain the origin.

MODIFIED POINCARÉ-BIRKHOFF THEOREM. *Let $T: A_1 \rightarrow T(A_1) \subset \mathbf{R}^2 \setminus \{O\}$ be an area-preserving homeomorphism. Suppose that the boundary circle C_1 is invariant under T . (In the degenerate case this means the origin is also the inner boundary of $T(A_1)$.) Moreover, suppose that T has a lifting \tilde{T} defined by (1.1) such that $g(r, \theta) < 0$ on $P^{-1}(C_2)$ and $\lim_{r \rightarrow R_1} \inf g(r, \theta) > 0$. Then \tilde{T} has at least two fixed points such that their images under P are two different fixed points of T .*

One asks naturally: what will happen if we allow the inner boundary also variant under T ? This problem is of importance because often the map T raised in a concrete problem does not possess the obvious invariant curve or fixed point. It is the aim of this paper to answer the above question. Our result is the following theorem.

Let A denote an annular region. The inner boundary C_1 and the outer boundary C_2 of A are two closed simple curves. By D_i we denote the open region bounded by C_i , $i = 1, 2$.

THEOREM 1. *Let $T : A \rightarrow T(A) \subset \mathbf{R}^2 \setminus \{O\}$ be an area-preserving homeomorphism. Suppose that*

- (1) *the inner boundary curve C_1 is star shaped about the origin;*
- (2) *T has a lifting \tilde{T} defined by (1.1) such that $g(r, \theta) > 0$ on $P^{-1}(C_1)$ and $g(r, \theta) < 0$ on $P^{-1}(C_2)$;*
- (3) *there exists an area-preserving homeomorphism $T_0: \bar{D}_2 \rightarrow \mathbf{R}^2$, which satisfies $T_0|_A = T$ and $O \in T_0(D_1)$.*

Then \tilde{T} has at least two fixed points such that their images under P are two different fixed points of T .

We have proved this theorem for the case that A is the standard annulus $A_0: R_1^2 \leq x^2 + y^2 \leq R_2^2$, in a previous paper [6]. The present more general form is necessary for certain applications, as is shown in [5]. The condition (1) of the theorem is crucial for our proof. However, we doubt its necessity for the theorem. As to the condition (3), its necessity can be seen from the following simple example.

Let T be a mapping from the annulus $2^{-2} \leq x^2 + y^2 \leq 2^2$ into $\mathbf{R}^2 \setminus \{O\}$ such that T has a lifting defined by $r^* = \sqrt{r^2 + 1}$, $\theta^* = \theta + 1 - r$. It is clear that T is uniquely determined by \tilde{T} . Furthermore, T is a homeomorphism which preserves the element of area $dx dy = r dr d\theta$. We have $g(r, \theta) = 1 - r$ in this example, which is positive on $r = \frac{1}{2}$ and negative on $r = 2$. Therefore, all assumptions except (3) of Theorem 1 are satisfied. However, \tilde{T} has no fixed point since $r^* > r$ for any r .

2. Proof of Theorem 1. The idea of our proof is very simple. We want to construct a new mapping $T_2: \bar{D}_2 \setminus \{O\} \rightarrow \mathbf{R}^2 \setminus \{O\}$ which possesses the following properties:

- (a) T_2 is an area-preserving homeomorphism.
- (b) The restriction of T_2 on A agrees with T , i.e. $T_2|_A = T$.
- (c) The origin is the inner boundary of $T_2(\bar{D}_2 \setminus \{O\})$.

(d) T_2 has a lifting \tilde{T}_2 defined by

$$r^* = f_2(r, \theta), \quad \theta^* = \theta + g_2(r, \theta),$$

such that $f_2 = f$, $g_2 = g$ on $P^{-1}(A)$, i.e. $\tilde{T}_2|_{P^{-1}(A)} = \tilde{T}$ and $g_2(r, \theta) \geq a > 0$ on $P^{-1}(D_1 \setminus \{O\})$.

If such T_2 exists, we see from (a), (c), (d) and (2) of Theorem 1 that T_2 and its lifting \tilde{T}_2 satisfy all assumptions of the Modified Poincaré-Birkhoff Theorem. It follows that \tilde{T}_2 has at least two fixed points such that their images under P are two different fixed points of T_2 . However, \tilde{T}_2 cannot have any fixed point in $P^{-1}(D_1 \setminus \{O\})$, since $g_2(r, \theta) > 0$ on this region by (d). Therefore, all fixed points of \tilde{T}_2 must be contained in $P^{-1}(A)$. As $\tilde{T}_2|_{P^{-1}(A)} = \tilde{T}$ and $T_2|_A = T$, we conclude at once that \tilde{T} has at least two fixed points such that their images under P are two different fixed points of T , thus completing the proof.

We now proceed to construct the mapping T_2 .

LEMMA 1. *There exists an area-preserving homeomorphism $T_1: \bar{D}_2 \rightarrow \mathbf{R}^2$ such that $T_1|_A = T$ and $T_1(O) = O$.*

PROOF. Let T_0 be the mapping given by (3) of the theorem. Then we have $p = T_0^{-1}(O) \in D_1$. Choose points $O = P_0, P_1, P_2, \dots, P_n = P$ in the segment \overline{OP} , such that the length $2d_i$ of the subsegment $\overline{P_{i-1}P_i}$ ($i = 1, 2, \dots, n$) is less than $d = \text{distant}(\overline{OP}, C_1)$. By (1) of the theorem, we have $d > 0$. For each $i \in \{1, 2, \dots, n\}$ we construct an area-preserving homeomorphism S_i as follows. Let O_i denote the middle point of $\overline{P_{i-1}P_i}$. Choose any polar coordinate system (ρ_i, φ_i) in \mathbf{R}^2 which has O_i as its pole. With this coordinate system define $S_i: \mathbf{R}^2 \rightarrow \mathbf{R}^2$ by

$$\rho_i^* = \rho_i, \quad \varphi_i^* = \varphi_i + h_i(\rho_i).$$

Here, $h_i(\rho_i)$ is a C^∞ function satisfying that $h_i(\rho_i) = 0$ for $\rho_i \geq \frac{1}{2}(d_i + d)$, $h_i(\rho_i) = \pi$ for $0 < \rho_i \leq d_i$. It is clear from this definition that $S_i|_A = \text{identity}$ and $S_i(P_{i-1}) = P_i$. It is also easy to check that S_i is an area-preserving diffeomorphism. Now, let $S = S_n \circ S_{n-1} \circ \dots \circ S_1$. Then S is an area-preserving diffeomorphism satisfying that $S|_A = \text{identity}$ and $S(O) = P$. Finally, define $T_1 = T_0 \circ S$. We see that T_1 is an area-preserving homeomorphism since it is a composition of such mappings. Moreover, $T_1|_A = T_0 \circ S|_A = T_0|_A = T$, and $T_1(O) = T_0(S(O)) = T_0(P) = T_0(T_0^{-1}(O)) = O$. This completes the proof of Lemma 1.

Now, let \tilde{T}_1 be a lifting of the restriction of T_1 on $\bar{D}_2 \setminus \{O\}$. Suppose \tilde{T}_1 is defined by

$$(2.1) \quad r^* = f_1(r, \theta), \quad \theta^* = \theta + g_1(r, \theta).$$

Since $T_1|_A = T$, we can choose \tilde{T}_1 such that

$$(2.2) \quad f_1(r, \theta) = f(r, \theta), \quad g_1(r, \theta) = g(r, \theta) \quad \text{on } P^{-1}(A),$$

i.e. $\tilde{T}_1|_{P^{-1}(A)} = \tilde{T}$. By (2) of Theorem 1, we have $g_1 = g > 0$ on $P^{-1}(C_1)$. Since $g_1(r, \theta)$ is 2π -periodic in θ there exists a neighborhood U of C_1 such that

$$(2.3) \quad g_1(r, \theta) \geq a > 0 \quad \text{on } P^{-1}(U),$$

for some constant a . Let C'_1 be any smooth closed simple curve in $\bar{D}_1 \cap U$ which is strictly star shaped about the origin. By *strictly star shaped* we mean that any ray from the origin hits C'_1 at just one point and nontangentially. The existence of such curves is guaranteed by (1) of the theorem. Let (x, y) be any point in $\mathbf{R}^2 \setminus \{O\}$. The

ray starting from the origin and passing (x, y) will hit C'_1 at a unique point $P(x, y)$. Define a function $\alpha: \mathbf{R}^2 \setminus \{O\} \rightarrow \mathbf{R}_+$ by $\alpha(x, y) = 1/|P(x, y)|$, where $|\cdot|$ denotes the Euclidean norm of \mathbf{R}^2 . Because C'_1 is smooth and strictly star shaped the function α is continuously differentiable. Note that for any $k > 0$ we have

$$\alpha(kx, ky) = \alpha(x, y).$$

Differentiating with respect to k and letting $k = 1$, we get

$$(2.4) \quad x \frac{\partial \alpha}{\partial x} + y \frac{\partial \alpha}{\partial y} = 0.$$

Define $\lambda(x, y) = \alpha(x, y)\sqrt{x^2 + y^2}$ for $x^2 + y^2 \neq 0$, and $\lambda(0, 0) = 0$. It is easy to see that $\lambda^{-1}(1) = C'_1$, and for any $\mu > 0$, $\lambda^{-1}(\mu) = C'_\mu$ is a similar curve of C'_1 about the origin. We can also easily see that $\lambda^{-1}[0, 1) = D'_1$ is just the open region bounded by C'_1 . Choose $\epsilon > 0$ so small that

$$(2.5) \quad \lambda^{-1}(1 - \epsilon, 1) \subset U.$$

For $\mu > 0$, set

$$(2.6) \quad \begin{aligned} \beta(\mu) &= \sup\{x^2 + y^2 \mid (x, y) \in C'_\mu\}, \\ \gamma(\mu) &= \sup\{|g_1(r, \theta)| \mid (r, \theta) \in P^{-1}(C'_\mu)\}. \end{aligned}$$

LEMMA 2. *There exists an area-preserving homeomorphism R of $\mathbf{R}^2 \setminus \{O\}$ onto $\mathbf{R}^2 \setminus \{O\}$ such that R has a lifting \tilde{R} defined by*

$$(2.7) \quad r^* = \phi(r, \theta), \quad \theta^* = \theta + \psi(r, \theta),$$

which satisfies

$$(2.8) \quad \phi(r, \theta) = r, \quad \psi(r, \theta) = 0 \quad \text{for } (r, \theta) \in P^{-1}(\mathbf{R}^2 \setminus D'_1),$$

$$(2.9) \quad \psi(r, \theta) \geq 0 \quad \text{for all } (r, \theta) \in H,$$

$$(2.10) \quad \psi(r, \theta) \geq \gamma(\mu) + a \quad \text{for } (r, \theta) \in P^{-1}(C'_\mu) \text{ and } \mu \in (0, 1 - \epsilon].$$

PROOF. Let $F : (0, +\infty) \rightarrow \mathbf{R}$ be a C^∞ function satisfying that $F(u) = 0$ for $u \geq 1$, $F(u) > 0$ for $u \in (1 - \epsilon, 1)$, and $F(u) \geq u^{-1}\beta(u)[\gamma(u) + a]$ for $u \in (0, 1 - \epsilon]$. Set

$$H(x, y) = \int_0^{\lambda(x, y)} F(u) du.$$

Consider the following Hamiltonian system defined on $\mathbf{R}^2 \setminus \{O\}$:

$$(2.11) \quad \begin{cases} \dot{x} = -\frac{\partial H}{\partial y} \equiv -F(\lambda(x, y)) \left[\frac{y\alpha(x, y)}{\sqrt{x^2 + y^2}} + \frac{\partial \alpha}{\partial y} \sqrt{x^2 + y^2} \right], \\ \dot{y} = \frac{\partial H}{\partial x} \equiv F(\lambda(x, y)) \left[\frac{x\alpha(x, y)}{\sqrt{x^2 + y^2}} + \frac{\partial \alpha}{\partial x} \sqrt{x^2 + y^2} \right]. \end{cases}$$

By the definition of $F(u)$ and $\lambda(x, y)$, the set $\lambda^{-1}[1, \infty) = \mathbf{R}^2 \setminus D'_1$ consists of all stationary points of the system (2.11). Also, we can easily verify that $\lambda(x, y)$ is a first integral of (2.11) and for each $\mu \in (0, 1)$, C'_μ is an integral curve of (2.11).

Let $(\bar{x}(t, x, y), \bar{y}(t, x, y))$ denote the solution of (2.11) that satisfies the initial condition $\bar{x}(0) = x, \bar{y}(0) = y$. We define the mapping $R : \mathbf{R}^2 \setminus \{O\} \rightarrow \mathbf{R}^2 \setminus \{O\}$ by

$$R(x, y) = (\bar{x}(1, x, y), \bar{y}(1, x, y)).$$

Note that R is the time-one mapping of the Hamiltonian flow induced by (2.11). Therefore R is an area-preserving homeomorphism, and for $(x, y) \in C'_\mu, R(x, y) \in C'_\mu$. In particular, we have $R(x, y) = (x, y)$ for $(x, y) \in \mathbf{R}^2 \setminus D'_1$.

Now, take the transformation $x(t) = r(t) \cos \theta(t), y(t) = r(t) \sin \theta(t)$ for (2.11). The resulting equations for $r(t), \theta(t)$ are

$$(2.12) \quad \begin{cases} \dot{\gamma} = -\cos \theta \frac{\partial H}{\partial y} + \sin \theta \frac{\partial H}{\partial x}, \\ \dot{\theta} = \frac{1}{\gamma} \left(\cos \theta \frac{\partial H}{\partial x} + \sin \theta \frac{\partial H}{\partial y} \right). \end{cases}$$

Employing (2.4), the second equation of (2.12) can be reduced to

$$(2.13) \quad \dot{\theta} = \frac{\lambda(x, y)F(\lambda(x, y))}{x^2 + y^2}.$$

Let $(\bar{r}(t, r, \theta), \bar{\theta}(t, r, \theta))$ denote the solution of (2.12) which satisfies $\bar{r}(0) = r, \bar{\theta}(0) = \theta$. Set $\phi(r, \theta) = \bar{r}(1, r, \theta), \psi(r, \theta) = \bar{\theta}(1, r, \theta) - \theta$. Then we see that R has a lifting \bar{R} which is defined by

$$r^* = \phi(r, \theta), \quad \theta^* = \theta + \psi(r, \theta).$$

It is clear that $\phi(r, \theta) = r, \psi(r, \theta) = 0$ on $P^{-1}(\mathbf{R}^2 \setminus D'_1)$ where every point is a stationary point of (2.12). It is also easy to see that $\lambda \circ P(r, \theta)$ is a first integral of (2.12). Thus we have

$$\lambda \circ P[\bar{r}(t, r, \theta), \bar{\theta}(t, r, \theta)] \equiv \mu, \quad \text{for } (r, \theta) \in P^{-1}(C'_\mu).$$

It follows from (2.13) that

$$(2.14) \quad \dot{\bar{\theta}}(t, r, \theta) = \mu F(\mu) / \bar{r}^2(t, r, \theta), \quad \text{for } (r, \theta) \in P^{-1}(C'_\mu).$$

For any $(r, \theta) \in H$, we see from (2.14) that $\dot{\bar{\theta}}(t, r, \theta) \geq 0$. Hence,

$$\psi(r, \theta) = \bar{\theta}(1, r, \theta) - \theta \geq 0, \quad \text{for each } (r, \theta) \in H.$$

For $(r, \theta) \in P^{-1}(C'_\mu)$ and $\mu \in (0, 1 - \epsilon]$, (2.14) and (2.6) yield

$$\dot{\bar{\theta}}(t, r, \theta) \geq \beta(\mu)[\gamma(\mu) + a] / \bar{r}^2(t, r, \theta) \geq \gamma(\mu) + a.$$

Integrating on $(0, 1)$, we get

$$\psi(r, \theta) = \bar{\theta}(1, r, \theta) - \theta \geq \gamma(\mu) + a, \quad \text{for } (r, \theta) \in P^{-1}(C'_\mu), \mu \in (0, 1 - \epsilon].$$

This completes the proof of Lemma 2.

Now let T_1 and R be the mappings given by Lemma 1 and Lemma 2 respectively. We define $T_2 = T_1 \circ R$ and verify that T_2 possesses the asserted properties (a)–(d). Since the verification of (a)–(c) is trivial, it suffices to show that (d) is satisfied by T_2 .

Let \tilde{T}_1 be the lifting of $T_1|(\bar{D}_2 \setminus \{O\})$ defined by (2.1). Let \tilde{R} be the lifting of R given by Lemma 2. Note that $T_2 = T_1 \circ R$ is a lifting of T_2 . Suppose that \tilde{T}_2 is

given by

$$r^* = f_2(r, \theta), \quad \theta^* = \theta + g_2(r, \theta).$$

By (2.1) and (2.7),

$$\begin{aligned} f_2(r, \theta) &= f_1(\phi(r, \theta), \theta + \psi(r, \theta)); \\ g_2(r, \theta) &= \psi(r, \theta) + g_1(\phi(r, \theta), \theta + \psi(r, \theta)). \end{aligned}$$

From (2.8) and (2.2) it follows that

$$f_2(r, \theta) = f(r, \theta), \quad g_2(r, \theta) = g(r, \theta), \quad \text{for } (r, \theta) \in P^{-1}(A).$$

Suppose now $(r, \theta) \in P^{-1}(D_1 \setminus \{O\})$. If $(r, \theta) \in P^{-1}(U \setminus D_1')$, then

$$g_2(r, \theta) = g_1(r, \theta) \geq a$$

by (2.8) and (2.3). If $(r, \theta) \in P^{-1}(\lambda^{-1}(1 - \epsilon, 1)) \subset P^{-1}(U)$, noting that $\tilde{R}(r, \theta) \in P^{-1}(\lambda^{-1}(1 - \epsilon, 1))$ too, we have

$$g_2(r, \theta) \geq g_1(\phi(r, \theta), \theta + \psi(r, \theta)) \geq a$$

by (2.9) and (2.3). Finally, if $(r, \theta) \in P^{-1}(\lambda^{-1}(0, 1 - \epsilon))$ then $(r, \theta) \in P^{-1}(C'_\mu)$ for some $\mu \in (0, 1 - \epsilon)$. It follows from (2.10) and (2.6) that

$$g_2(r, \theta) \geq \gamma(\mu) + a + g_1 \circ \tilde{R}(r, \theta) \geq a.$$

Here we have made use of the fact that $\tilde{R}(r, \theta) \in P^{-1}(C'_\mu)$ for $(r, \theta) \in P^{-1}(C'_\mu)$. In summary, we have in any case that $g_2(r, \theta) \geq a > 0$ for $(r, \theta) \in P^{-1}(D_1 \setminus \{O\})$. Thus (d) is satisfied.

The proof of Theorem 1 is complete.

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